Qualitative Criteria for Fuzzy Linguistic Summaries with Absolute Linguistic Expressions

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1 Introduction

Fuzzy linguistic summarization [8] enables providing concise and easy-to-interpret descriptions about large datasets. Its main aim is to aggregate and translate numeric observations into natural-language sentences using linguistic expressions. Countless examples confirm the usefulness of fuzzy linguistic summaries in practical applications, in particular those based on sensor-collected data, when their analysis is too complex or time-consuming for experts.

This contribution builds upon previous achievements in the theories of generalized and intermediate quantifiers [1], and the evaluative linguistic expressions [2]. When constructing fuzzy linguistic summaries, one can distinguish relative expressions such as *low, high, medium, short*, etc. or absolute ones, e.g., *around 20* [9]. In the majority of the related works, the relative expressions are sufficient. However, as observed for the clinical setting, there are well-established norms arising from the medical guidelines that need to be acknowledged. Absolute expressions are of great importance in medical applications, where many indicators have numerical standards, such as blood or heart rate tests (e.g., 'About 100 patients with high blood pressure have a pulse of around 90.'). Thus, there is a need to consider absolute linguistic expressions, which are usually represented as **unimodal membership functions**, and, in our opinion, the properties of such summaries have not been studied intensely so far.

In this work, we study the antonym property of fuzzy linguistic summaries with absolute linguistic expressions. First, we briefly review qualitative evaluation criteria with a particular focus on the degree of truth (as baseline) and the degrees of imprecision and specificity. Next, we consider the property of antonym and investigate its adequacy for the selected criteria.

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2 Fuzzy linguistic summaries and their qualitative criteria

First, let us recall the definition of fuzzy linguistic summary (see [6]). For the given set of objects Y, let $A = \{A_1, \ldots, A_m\}$ denote a set of attributes, where A_i is a function $A_i \colon Y \to X_i$, $i = 1, \ldots, m$ and $X_i \neq \emptyset$. Let $L(Y, A_i)$ be a set of linguistic expressions for a given $i \in \{1, \ldots, m\}$ defined as follows $L(Y, A_i) = \{l_1^{A_i}, \ldots, l_{k_i}^{A_i}\}$ and defines the formulation of summaries in natural language. Also, let $k_i = |L(Y, A_i)|$ for given $i \in \{1, \ldots, m\}$. Moreover, let $\mathcal{V} = \{V_{i,k} \mid V_{i,k} \colon A_i(Y) \to [0,1], i = 1, \ldots, m, \ k = 1, \ldots, k_i\}$. With this notation, we have the following definition.

Definition 1 ([6]) *For Y being a set of objects, let $S = (A, L, (\mathcal{P}, \diamond), (\mathcal{R}, \star), Q)$ be a 5-tuple (quintuple) such that*

(i) $A = \{A_1, \ldots, A_m\}$ is a set of attributes $A_i \colon Y \to X_i$, $i = 1, \ldots, m$, where $X_i \neq \emptyset$,

(ii) $L = \{L_1, \ldots, L_m\}$ is the set of linguistic expressions of sets $L_i(Y, A_i) = \{l_1^{A_i}, \ldots, l_{k_i}^{A_i}\}$, $i = 1, \ldots, m$,

(iii) $\mathcal{P} \subset \mathcal{V}$ is a family of summarizers, $\mathcal{R} \subset \mathcal{V}$ is a family of qualifiers, and they satisfy

$$\exists_{i \in \{1,\ldots,m\}} \exists_{k \in \{1,\ldots,k_i\}} V_{i,k} \in \mathcal{P} \Rightarrow \forall_{k \in \{1,\ldots,k_i\}} V_{i,k} \notin \mathcal{R}$$

and

$$\exists_{i \in \{1,\ldots,m\}} \exists_{k \in \{1,\ldots,k_i\}} V_{i,k} \in \mathcal{R} \Rightarrow \forall_{k \in \{1,\ldots,k_i\}} V_{i,k} \notin \mathcal{P},$$

(iv) $Q \colon Z \to [0,1]$ is a linguistic quantifier and $Z \in \{\mathbb{R}^+, [0,1]\}$.

(v) $\diamond, \star$ are aggregation functions that can be conjunctive or disjunctive. If $card(\mathcal{P}) = 1$ or $card(\mathcal{R}) = 1$, then we assume $\diamond \equiv id$ or $\star \equiv id$, respectively.

*Then S is called a **fuzzy linguistic summary** of the following general form: Q $R_1 \star \ldots \star R_k$ y's are $P_1 \diamond \ldots \diamond P_l$ with $\star, \diamond \in \{AND,\ OR\}$.*

Note, that linguistic quantifiers can be seen as monadic **L**-fuzzy quantifiers of the type $< 1^n, 1 >$ [4]. For $\mathcal{P}, \mathcal{R} \subset \mathcal{V}$, we have that $\mathcal{P} = \{P_1, \ldots, P_l\}$ and $\mathcal{R} = \{R_1, \ldots, R_p\}$. For each object $y_i \in Y, i = 1, \ldots, n$, let $x_i = A_s(y_i)$ for $A_s \in A, s = 1, \ldots, m$, we take

$$R(x_i) = \begin{cases} R_1(x_i), & p = 1, \\ F_1(R_1(x_i), \ldots, R_p(x_i)), & p > 1, \end{cases} \quad \text{and} \quad P(x_i) = \begin{cases} P_1(x_i), & l = 1, \\ F_2(P_1(x_i), \ldots, P_l(x_i)), & l > 1, \end{cases}$$

where $F_1 \colon [0,1]^p \to [0,1]$, $F_2 \colon [0,1]^l \to [0,1]$ can be conjunctions or disjunctions. Now, let $j \in \mathbb{N}$ and $\mathcal{T} = \{T_i | T_i \colon \mathcal{S} \to [0,1], i = 1, \ldots, j\}$ be the family of functions, where $\mathcal{S}$ is a family of fuzzy summaries S. $\mathcal{T}$ is called **j-tuple of qualitative criteria** of S. Here, we consider few qualitative criteria starting with the most important one - **the degree of truth** introduced by Zadeh [9]. Following that approach, we present the following formula:

$$T_1(S) = \begin{cases} Q\left(\frac{\sum_{i=1}^n C(P(x_i), R(x_i))}{\sum_{i=1}^n R(x_i)}\right), & Q \text{ is relative}, \\ Q\left(n \cdot \frac{\sum_{i=1}^n C(P(x_i), R(x_i))}{\sum_{i=1}^n R(x_i)}\right), & Q \text{ is absolute}, \end{cases} \tag{1}$$

where S is a fuzzy linguistic summary, C is a conjunctive aggregation function.
Another criteria are following (we use the above notation for FLSs):

(i) **degree of imprecision** ([5]). Let $A \in \mathcal{F}(X)$, where $\mathcal{F}(X)$ denotes a set of all fuzzy sets of $X \neq \emptyset$. The degree of imprecision of A is given by $im(A) = \frac{card\{x \in X:\, A(x)>0\}}{card X}$. The degree of imprecision of a summarizer is given by $im_P = \sqrt[l]{\prod_{k=1}^{l} im(P_k)}$, where $P_k \in \mathcal{P},\ k = 1, \ldots, l$. Analogously, it is defined for the qualifier: $im_R = \sqrt[p]{\prod_{k=1}^{l} im(R_i)}$, where $R_k \in \mathcal{R},\ k = 1, \ldots, p$. Then the degree of imprecision of the FLS is given by

$$T_2(S) = w_R \cdot im_R + w_P \cdot im_P + w_Q \cdot im(Q), \tag{2}$$

where $w_P, w_Q, w_R \geq 0$ and $w_P + w_R + w_Q = 1$.

(ii) **the degree of specificity** ([7, 5]). Here, let us first recall a measure of specificity ([8]). A function $Sp\colon \mathcal{F}(X) \to [0,1]$ is called a measure of specificity if we have
- $Sp(A) = 1 \Leftrightarrow A = \{x\},\ \ Sp(\emptyset) = 0,$,
- $\frac{\partial Sp(A)}{\partial a_1} > 0$ and $\frac{\partial Sp(A)}{\partial a_j} \leq 0$ for all $j \geq 2$, where $A \in \mathcal{X}$ and a_j is the j-th largest membership grade in A.

Yager proposed it as follows $Sp(A) = \int_0^{\alpha_{max}} \frac{1}{card(A_\alpha)} d\alpha$, where α_{max} is the largest membership grade in A, A_α is the α-level set of A, (i.e. $A_\alpha = \{x : A(x) \geq \alpha\}$, $card(A_\alpha)$ is the number of elements in A_α. Here, we follow [5] and use $Sp(A) = \alpha_{max} -$ area under A. Now, the degree of specificity of the FLS is given by

$$T_3(S) = w_R Sp(R) + w_P Sp(P) + w_Q Sp(Q), \tag{3}$$

where $w_P, w_Q, w_R \geq 0$ and $w_P + w_R + w_Q = 1$ and Sp is the measure of specificity.

3 Examples of FLSs with absolute quantifiers

In this section, we present some examples of FLSs with absolute quantifiers and analyze the qualitative criteria for them taking into account FLSs with negations. We focus on the property of antonyms ([3]). It is actually a special case of a double negation property. However, here the only kind of negation considered for a linguistic quantifier is its antonym.

Definition 2 ([3]) *Let Q be an absolute quantifier, then its antonym quantifier is given by $Q_{Ant}(k) = Q(n-k)$, $k = 1, \ldots, n$ and n is a number of objects. The* **antonym property** *is satisfied if for a given criterion T we have*

$$T(S) = T(S_{neg}),$$

where $S_{neg} = (A, L, (\{\neg P_1, \ldots, \neg P_l\}, \diamond_D), (\mathcal{R}, \star), Q_{Ant})$ and $\diamond_D$ is a dual operator of $\diamond$.

For absolute linguistic quantifiers, we can consider their antonyms for 3 types of them: increasing, decreasing, and unimodal - which make a large part of all such quantifiers. For instance, if we know n - the number of objects, then the antonym of a quantifier 'about x' is 'about $n - x$'.

Example 3 *Let us focus on degrees of truth, imprecision, and specificity. Note that for a sentence $S =$*'About 10 employees earn low salary', *$S_{neg} =$*'About $n-10$ employees earn **not** low salary', *where n is the number of all employees and* '**not** low' *is a negation of a summarizer. Here, we analyze the antonym property for the following FLSs:*

(i) 'About 10 employees earn low salary.' *Here, this property is satisfied in the case of a degree of truth. For the degrees of imprecision and specificity, it is not satisfied in general. In most cases, this depends on the shape of summarizers.*

(ii) 'About 10 employees with small experience are about 30 years old.' *For this sentence, for the degree of truth, the antonym property is satisfied in some cases - it is related to the choice of the appropriate conjunction in the formula* (1). *Similarly as before, this property is not satisfied for the other two criteria.*

(iii) 'About 10 employees earn low salary and have small experience.' *Again, for the degree of truth, the antonym property is satisfied, while for the degrees of imprecision and specificity, it is not. It can be satisfied only for specific summarizers (it depends on their (non)symmetric shape with respect to the number of objects).*

In our contribution, we will analyze such FLSs with absolute expressions and comment on when the antonym property is satisfied and what it depends on.

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