# Fuzzy–Probabilistic Inference Systems Based on Piecewise Linear Weighted Quantiles

CAO Nhung and HOLČAPEK Michal and VALÁŠEK Radek

*Institute for Research and Applications of Fuzzy Modeling, University of Ostrava*
*30. dubna 22, 701 03 Ostrava, Czech Republic*
*E-mail:* {nhung.cao, michal.holcapek, radek.valasek}@osu.cz

## 1 Introduction and motivation

Fuzzy–probabilistic IF–THEN rule based systems form a class of knowledge-based systems where the antecedents encode fuzzy information and the consequents model probabilistic uncertainty present in the data. By integrating fuzzy set theory with probability theory, these systems offer a unified framework for managing both types of uncertainty and have been extensively investigated in the context of rule construction and inference design (see, e.g., [1, 2]).

In this work, we use a specific form of the IF-THEN rules and the inference mechanism, which coincide with the so-called *quantile fuzzy transform* (or $L_1$-fuzzy transform) that has been introduced and extensively investigated in [3, 4]. Particularly, given a suitable fuzzy partition $\Delta = \{A_1, \dots, A_m\}$ of a universe $\mathbb{U}$ and a random variable $Y$ defined in a probability space $(\Omega, \mathcal{A}, P)$, the system takes the following formal form:

$$R_k : \text{IF } X \text{ is } A_k \text{ THEN } Y \sim Q_k(p), \quad k = 1 \dots, m, \tag{1}$$

where $Q_k(p)$ denotes the quantile function of the conditional distribution of $Y$ given $A_k$. A crucial step in constructing the IF-THEN rules is the estimation $Q_k(p)$ from data. In [3], the quantile functions are derived from data using $p$-weighted quantiles, which are traditionally computed by the linear programming. In our recent work [5], we introduced an alternative and computationally efficient method for evaluating weighted quantiles derived from the analyzes of the right derivative of the corresponding convex objective function (see (2)). Nevertheless, prior research indicates that, despite their computational efficiency, classical weighted quantiles may be inadequate for accurately estimating the local positions of the output quantiles over the fuzzy inputs.

To address the limitation of the scalar weighted quantiles, we proposed to replace them by quantile piecewise linear functions [6], which better fit the output quantiles. This short paper presents a modification of the original method to enhance its applicability to forecasting tasks and includes also a comparison with the scalar weighted quantile approach.

## 2 Outline of the studied approach

Let us consider the fuzzy-probabilistic IF–THEN rules given in (1). The fuzzy sets $A_k$ in the antecedents form a fuzzy partition $\Delta$ on $\mathbb{U}$ with nodes $c_1, \dots, c_m$ in which each $A_k$ attains its

**Acknowledgement** The study is supported by the project “Research of Excellence on Digital Technologies and Wellbeing CZ.02.01.01/00/22_008/0004583” which is co-financed by the European Union.



https://www.doi.org/10.15452/978-80-7599-515-5.2026.04

maximum. The fuzzy partition covers $\mathbb{U}$, i.e., for any $x \in \mathbb{U}$, there exists $k = 1, \ldots, m$ with $A_k(x) > 0$. Uniform or generalized fuzzy partitions are usually employed.

The empirical quantile functions $Q_k(p)$ in the consequents are estimated from data and represent the conditional distribution of $Y$ given $A_k$. Particularly, let $\{(x_i, y_i)\}_{i=1}^n \subseteq \mathbb{U} \times \mathbb{R}$ be data, and let $\Delta$ be a fuzzy partition. For each $A_k$ and $p \in [0,1]$, the *p-weighted quantile* $Q_k(p)$ is a real number $z$ such that minimizes the following functional

$$\Phi_p^k(z) = \sum_{i=1}^n \rho_p(y_i - z)A_k(x_i), \quad \rho_p(u) = \begin{cases} p|u|, & u > 0, \\ (1-p)|u|, & u \leq 0. \end{cases} \tag{2}$$

Observe that $A_k(x_i)$ serves as a weight for $y_i$; the closer $x_k$ is to $c_k$, the higher the resulting degree. The vector $[Q_1(p), \ldots, Q_m(p)]$ is known as the *direct Quantile Fuzzy transform* [3, 4]. Note that the weighted quantiles $Q_k(p)$ can be computed via linear programming [3] or the direct derivative-based method [5, 6]. The inference mechanism, which estimates the quantile function $Q_x$ for any $x \in \mathbb{U}$, is defined as a weighted combination of quantiles $Q_k$ (known as the *inverse QF-transform*):

$$Q_x(p) = \frac{\sum_{k=1}^m A_k(x) Q_k(p)}{\sum_{k=1}^m A_k(x)}. \tag{3}$$

In the following part, we generalize the scalar $p$-weighted quantiles to the piecewise linear $p$-weighted quantile functions associated with a *centering initial point* $(c_k, z_p) \in \mathbb{U} \times \mathbb{R}$ and extend the system introduced in (1). In practice, the centering initial point $(c_k, z_p)$ corresponds to the $k$-th node in $\mathbb{U}$ and the $p$-weighted quantile $z_p = Q_k(p)$ is determined by $A_k \in \Delta$.

Let $p \in [0,1]$, $A_k \in \Delta$, and fix $(c_k, z_p) \in \mathbb{U} \times \mathbb{R}$. For any real numbers $a_1, a_2$, define the piecewise linear function

$$Q_{k,(a_1,a_2)}(p, x) = \begin{cases} a_1(x - c_k) + z_p, & x \leq c_k, \\ a_2(x - c_k) + z_p, & x \geq c_k. \end{cases} \tag{4}$$

Define the objective functional

$$\Phi_p^k(a_1, a_2) = \sum_{i=1}^n \rho_p(y_i - Q_{k,(a_1,a_2)}(p, x_i))A_k(x_i). \tag{5}$$

Then the *piecewise linear p-weighted function quantile associated with* $(c_k, z_p)$ is $Q_{k,(\hat{a}_1,\hat{a}_2)}(p, \cdot) : \mathbb{U} \to \mathbb{R}$, where $(\hat{a}_1, \hat{a}_2)$ is any minimizer of $\Phi_p^k$ over all $(a_1, a_2) \in \mathbb{R}^2$. Similarly to the scalar case, the piecewise linear $p$-weighted quantile is not uniquely defined. The following lemma guarantees that such a quantile always exists.

**Lemma 1** *For each $A_k \in \Delta$ and each centering initial point $(c_k, z_p) \in \mathbb{U} \times \mathbb{R}$, there exist a pair $(\hat{a}_1, \hat{a}_2)$ minimizing $\Phi_p^k$ over all $(a_1, a_2) \in \mathbb{R}^2$.*

Observe that we put if $z_p = Q_k(p)$, then $Q_k(p) = Q_{k,(\hat{a}_1,\hat{a}_2)}(p, c_k)$. Hence,

$$Q_{k,(\hat{a}_1,\hat{a}_2)}(p, c_k) \leq Q_{k,(\hat{a}_1,\hat{a}_2)}(p', c_k), \quad \text{whenever } p \leq p'.$$

Obviously, the same property is not true for an arbitrary $x \in \mathbb{U}$, i.e., $Q_k(\cdot, x) = Q_{k,(\hat{a}_1,\hat{a}_2)}(\cdot, x)$ is not a quantile function from its definition in general. To ensure that $Q_k(\cdot, x)$ is a quantile function for any $x \in \mathbb{U}$, we can proceed the following correction procedure:

$$Q_k^*(p, x) = \sup\{Q_k(q, x) \mid q \leq p\}, \quad p \in [0,1].$$

Note that similarly, we can use the infimum or a mixture of both procedures with an initial $p_0$. In addition, we consider only a finite number of $p$-levels, therefore, the computation of $Q_k^*(p, x)$ is fast in practice. Now, the IF-THEN rules specified in (1) can be extended as follows:

$$R_k : \text{IF } X \text{ is } A_k \text{ THEN } Y(x) \sim Q_k^*(p, x), \quad k = 1, \dots, m, \tag{6}$$

where $Q_k^*(p, x)$ expresses (locally) the conditional quantile function given $A_k$ at the point $x \in \mathbb{U}$. The inference mechanism for this new type of rule based system is given for each $x \in \mathbb{U}$ as follows

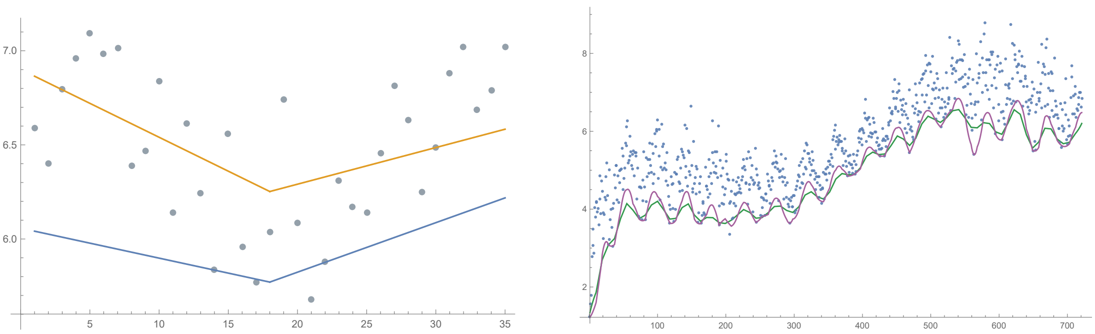
$$Q_x(p) = \frac{\sum_{k=1}^{m} A_k(x) Q_k^*(p, x)}{\sum_{k=1}^{m} A_k(x)}. \tag{7}$$

## 3 Application in time series analysis and forecasting

In this section, we illustrate the application of piecewise linear weighted quantiles within a fuzzy–probabilistic inference system (FPIS) using a synthetic time series of $n = 721$ observations $(x_i, y_i)$, generated by

$$y_i = g(x_i) + \varepsilon(x_i), \quad g(x) = \tfrac{1}{2}\sin\left(\tfrac{x}{7}\right) + \cos\left(\tfrac{x}{90}\right) + \ln(x + 1),$$

where $\varepsilon(x) \sim \mathcal{N}(0, \sigma(x))$, $\sigma(x) = 0.4 + \frac{1}{6}\sin\left(\frac{\pi x}{240}\right)$. We employ a uniform fuzzy partition $\Delta = \{A_1, \dots, A_{81}\}$ with an offset of 2 (the support of each fuzzy set is twice the shift length). In Fig. 1, panel (a) displays the conditional quantile functions $Q_k^*(p, x)$ given $A_{53}$ for fixed $p \in \{0.05, 0.75\}$, while panel (b) compares the resulting quantile functions of the rule systems (1) (shown in green) and (6) (shown in violet) along $\mathbb{U}$ for the fixed value $p = 0.05$. From a




(a) (b)

Figure 1: Illustration of piecewise linear and scalar weighted quantile based FPIS

visual comparison, one can observe that the application of piecewise linear weighted quantile functions yields more accurate estimations of the $p$-quantile functions along the universe $\mathbb{U}$. This observation is further supported by the Bhattacharyya distance between the theoretical and estimated (normally distributed) density functions, see Fig. 2.

Without going into details, the extended FPIS can be used to forecast probability distributions. To illustrate this, we employ a simple autoregressive model. First, we consider $\mathbb{U} = [0, T]$ and extend the fuzzy partition $\Delta$ to $\Delta^+$ by adding fuzzy sets $A_{m+1}, \dots, A_{m+v}$. Then, using the AR($\ell$) model, we estimate the parameters of the piecewise linear weighted quantile functions $(a_{1,m+i}, a_{2,m+i}, y_{m+i})$ for $i = 1, \dots, v$, and apply the inference mechanism given in (7) to obtain the quantile functions for future time points.

In Fig. 3(a), the forecasted $p$-quantiles for 45 future time points are shown, while Fig. 3(b) displays the ARMA models computed and trained on scalar quantiles in *Mathematica*. The forecast based on the FPIS outperforms almost a linear forecast provided by ARMA modeling.

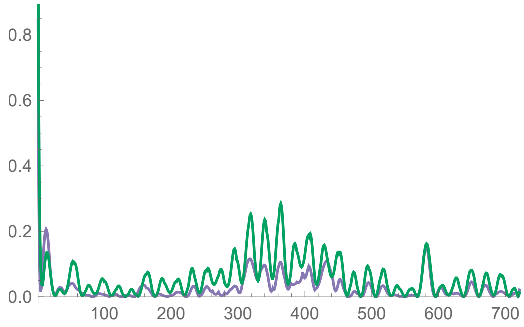
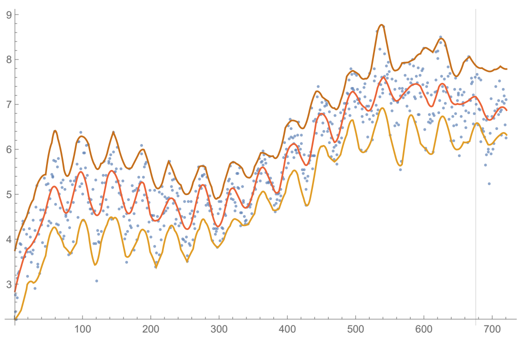
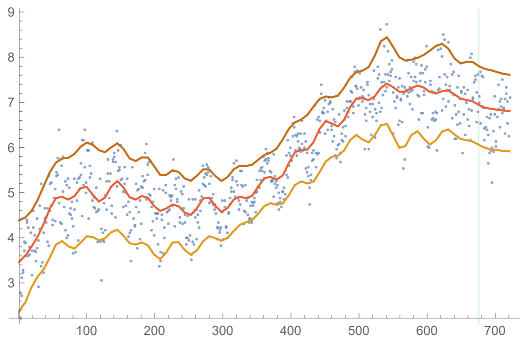



Figure 2: Bhattacharyya distance between the theoretical normal density function and its estimates obtained using the piecewise linear (violet) and scalar (green) weighted quantiles.



(a)




(b)

Figure 3: Forecasting for $p \in \{0.05, 0.5, 0.95\}$: (a) piecewise linear quantiles; (b) ARMA.

## References


[1] van den Berg, J., Kaymak, U., Almeida, R.J.: *Conditional density estimation using probabilistic fuzzy systems.* IEEE Trans. Fuzzy Syst. **20**(5) (2012) 869–882.

[2] Rudnik, K., Walaszek-Babiszewska, A.: *Probabilistic-fuzzy knowledge-based system for managerial applications.* Manag. Prod. Eng. Rev. **3** (2012) 49–61.

[3] Guerra, M.L., Sorini, L., Stefanini, L.: *Quantile and expectile smoothing based on $L_1$-norm and $L_2$-norm fuzzy transforms.* Int. J. Approx. Reason. **107** (2019) 17–43.

[4] Madrid, N.: *Significance measures for rules in probabilistic-fuzzy inference systems based on fuzzy transforms.* Fuzzy Sets Syst. **467** (2023) 108575.

[5] Holčapek, M., Cao, N., Valášek, R., Madrid, N., Tichý, T., Neděla, D.: *An exploration of the weighted quantile approach in probabilistic fuzzy inference.* Proc. IPMU2024 Short Papers, Lisbon (2024) 17–20.

[6] Cao, N., Holčapek, M., Valášek, R., Madrid, N.: *Probabilistic-fuzzy inference with piecewise linear quantile regression.* In: Int. Conf. on Modeling Decisions for Artificial Intelligence. Cham: Springer Nature Switzerland (2025) 214–226.