Fuzzy transform for operational matrices in fractional equations

PHAM Thi Minh Tam[1] and PERFILIEVA Irina[1]

[1] *University of Ostrava*
Institute for Research and Applications of Fuzzy Modeling
30. dubna 22, 701 03 Ostrava
Czech Republic
E-mail: thi.pham.s01,Irina.Perfilieva@osu.cz

Fractional differential and integral equations have attracted considerable attention in recent decades due to their ability to model the memory and hereditary properties of physical, biological, and engineering systems. Unlike classical (integer) derivatives, their fractional variants, such as Caputo or Riemann-Liouville derivatives, allow the system's dynamics to depend not only on its current state but also on its past behavior. As a result, fractional differential equations offer suitable models for describing anomalous diffusion and various processes with nonlocal effects. However, these models often lead to weakly singular kernels, which complicates analytical treatment and stimulates the development of various numerical approximation schemes. Below, we consider typical problems associated with fractional equations:

1. The fractional integral (Volterra) equation
$$y(t) = \mathcal{I}^{\beta}_{0+} y(t) + g(t), \tag{1}$$
2. The Initial Value Problem (IVP) with the Caputo fractional derivative
$$y(t) =_C \mathcal{D}^{\beta}_{0+} y(t) + g(t),\, y(0) = y_0, \tag{2}$$
3. The IVP with the Riemann-Liouville (R-L) fractional derivative
$$y(t) =_{RL} \mathcal{D}^{\beta}_{0+} y(t) + g(t),\, y(0) = y_0, \tag{3}$$

where y and g are the unknown and given functions, respectively. The operators $\mathcal{I}^{\beta}_{0+}$, ${}_C\mathcal{D}^{\beta}_{0+}$ and ${}_{RL}\mathcal{D}^{\beta}_{0+}$ are the fractional integral operator, the Caputo and the R-L operators of differentiation, respectively. For any suitable function y, the formal expressions of these operators are as follows:

$$\begin{aligned}
\mathcal{I}^{\beta}_{0+} y(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds, \\
{}_C\mathcal{D}^{\beta}_{0+} y(t) &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} y'(s) ds = \mathcal{I}^{1-\beta}_{0+}(y')(t), \\
{}_{RL}\mathcal{D}^{\beta}_{0+} y(t) &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left(\int_0^t (t-s)^{-\beta} y(s) ds \right) = \frac{d}{dt} \mathcal{I}^{1-\beta}_{0+} y(t),
\end{aligned} \tag{4}$$

where Γ denotes the gamma function.

Acknowledgement The work was partially supported by the GACR LA project 24-10177L "Fractional and fuzzy-fractional transport in disordered environments".

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`https://www.doi.org/10.15452/978-80-7599-515-5.2026.19`

Let us introduce a function space $C^{1,1-\beta}(0,1]$, which guarantees that all equations in (4), are well-defined [2, 3] and moreover, all these equations are solvable. The space $C^{1,1-\beta}(0,1]$ consists of continuous and continuously differentiable functions f in $(0,1]$ such that the following inequalities are valid:

$$\begin{cases} |f(t)| \le b, \\ |f'(t)| \le bt^{\beta-1}. \end{cases}$$

It is known that $C^{1,1-\beta}(0,1]$ is a Banach space where the norm is given by

$$\|f\|_{C^{1,1-\beta}(0,1]} = \sup_{t\in(0,1]} |f(t)| + \sup_{t\in(0,1]} t^{1-\beta}|f'(t)|.$$

It has been shown in [2] that the considered operators $\mathcal{I}^{\beta}_{0+}$, ${}_{C}\mathcal{D}^{\beta}_{0+}$, ${}_{RL}\mathcal{D}^{\beta}_{0+}$ are compact on $C^{1,1-\beta}(0,1]$. This fact guarantees that the approximation can be found in the form of a member of a convergent (to a solution) sequence.

We assume that $g \in C^{1,1-\beta}(0,1]$, so that the equations (1), (2), and (3) have unique solutions [2, 3]. However, in most cases, the solutions cannot be represented analytically. Therefore, approximation methods are required. With this idea in mind, we propose the application of the F-transformation methodology [5, 6, 7], which can reduce the original problems of (1), (2), and (3) to the corresponding systems of linear equations.

In this paper, we show that after applying the F-transform to the problems (1), (2) and (3), we obtain three systems of linear equations with the same matrix, which is either in direct or inverse form. We call this matrix as *operational matrix.*

First, let us give the necessary brief information about the F-transform method [4, 5].

Definition 1 (Fuzzy partition) *Let* $n \ge 3$. *Fuzzy sets* $A_0, A_1 \cdots, A_n : [0,1] \to [0,1]$, *identified with their continuous membership functions, constitute a fuzzy partition of* $[0,1]$ *with nodes* $0 = t_0 < t_1 < \cdots < t_n = 1$, *if they fulfill the following conditions:*

1. *(locality) For all* $k = 1, \cdots, n-1$, $A_k(t) = 0$ *if* $t \notin (t_{k-1}, t_{k+1})$; $A_0(t) = 0$ *if* $t \notin [0, t_1)$; $A_n(t) = 0$ *if* $t \notin (t_n, 1]$;
2. *(positiveness) For all* $k = 1, \cdots, n-1$, $A_k(t) > 0$ *if* $t \in (t_{k-1}, t_{k+1})$, $A_0(t) > 0$ *if* $t \in [0, t_1)$; $A_n(t) > 0$ *if* $t \in (t_{n-1}, t_n]$;
3. *(normality) For all* $k = 0, \cdots, n$, $A_k(t_k) = 1$.

Remark 1 *Let* $\gamma \ge 1$. *Fuzzy partition of* $[0,1]$ *with nodes* $0 = t_0 < t_1 < \cdots < t_n = 1$, *is said to be* graded *if*

$$t_i = \left(\frac{i}{n}\right)^{\gamma}, \, i = 0, \cdots, n. \tag{5}$$

Definition 2 (Direct and Inverse F^0-transform) *Let* $n \ge 2$, *and* $A_0, \ldots, A_n$ *be a fuzzy partition of* $[0,1]$ *with nodes* $t_0, t_1, \ldots, t_n$. *Let function* $f \in L^2[0,1]$. *Then,*

- *the (direct)* F^0*-transform of* f *is a vector of real numbers* $(F^0_0, \ldots, F^0_n)$ *such that*

$$F^0_k = \frac{\int_{t_{k-1}}^{t_{k+1}} f(t)A_k(t)dt}{\int_{t_{k-1}}^{t_{k+1}} A_k(t)dt},$$

where we set $t_{-1} = t_0 = 0$ *and* $t_{n+1} = t_n = 1$;

- *the inverse* F^0*-transform of* f *is a function* $\hat{f}^0_F \in C[0,1]$ *represented as*

$$\hat{f}^0_F(t) = \sum_{k=0}^{n} F^0_k A_k(t),$$

where $(F^0_0, \dots, F^0_n)$ *is the direct* F^0*-transform of* f*.*

Remark 2 *The direct* F^0*-transform of a function* $f \in L^2[0,1]$ *can be considered as result of a linear operator* $\mathbf{F}^0 : L^2[0,1] \to \mathbb{R}^{n+1}$ *so that*

$$\mathbf{F}^0[f] = (F^0_0, \dots, F^0_n).$$

Our goal is to apply the operator $\mathbf{F}^0$ to both sides of the equations (1), (2) and (3) (whose solutions are denoted by y) and transform each of them into a corresponding system of linear equations with respect to the unknown vector $\mathbf{F}^0[y]$, which is the direct F^0-transform of the function y.

In particular, the equation (1) will be rewritten as

$$\mathbf{F}^0[y] = \mathbf{F}^0[\mathcal{I}^{\beta}_{0+} y] + \mathbf{F}^0[g], \tag{6}$$

where $\mathbf{F}^0[y]$ and $\mathbf{F}^0[g]$ are vectors of F^0-transform components of y and g. To transform the equation (6) into a system of linear equations, we need to show that the vector $\mathbf{F}^0[\mathcal{I}^{\beta}_{0+} y]$ can be represented as $L\mathbf{F}^0[y]$, where L is the corresponding operational matrix that approximates the linear transformation of y performed by the integral operator $\mathcal{I}^{\beta}_{0+}$.

Further details will be given for the equation (6). Assume that $y \in C^{1,1-\beta}(0,1]$ is the exact solution of equation (1), and $0 < \beta < 1$. Assume also that all three functions from (1) are smooth and their second derivatives are bounded in the interval $(0,1)$. For the sake of simplicity we assume that limit initial values are: $y(0) = g(0) = 0$.

Choose $0 < \varepsilon < 1$, $n = \lceil \frac{1}{\varepsilon} \rceil$, $\gamma \geq \frac{1}{\beta}$, and arrange the nodes $t_1, \dots, t_n$ in $[0,1]$ according to (5). Suppose that the fuzzy partition of $[0,1]$ consists of the triangular-shaped fuzzy sets $A_0, A_1, \dots, A_n$ that fulfill the so called Ruspini condition (see [4] for the details).

Since zero initial value we may assume that $\mathbf{F}^0_0[y] = \mathbf{F}^0_0[g] = 0$. Therefore, the equation (6) can be considered with the reduced to n dimension with respect to the unknown F^0-transform components of y, which we denote as $\mathbf{Y} = (\mathbf{Y_1}, \dots, \mathbf{Y_n})$. Similarly the reduced vector of F^0-transform components of g will be denoted as $\mathbf{G} = (\mathbf{G_1}, \dots, \mathbf{G_n})$.

Theorem 3 *Let the assumptions given above be fulfilled. Then the* n*-dimensional vector of* F^0*-transform components* $\mathbf{F}^0[\mathcal{I}^{\beta}_{0+} y]$ *can be represented as* $\underline{L}^{\beta}\mathbf{Y}$*, where* $\underline{L}^{\beta}$ *is the operational matrix, corresponding to the integral operator* $\mathcal{I}^{\beta}_{0+}$*, and*

$$\underline{L}^{\beta} = \tfrac{1}{\Gamma(\beta)} \begin{pmatrix} a^{\beta}_{1,1} & 0 & \cdots & 0 \\ a^{\beta}_{2,1} & a^{\beta}_{2,2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a^{\beta}_{n,1} & a^{\beta}_{n,2} & \cdots & a^{\beta}_{n,n} \end{pmatrix},$$ *where we use the following notation:*

- $\Lambda_{k,l} = k - l$, $k, l \in \mathbb{N}$;
- *for* $k = 1, \dots, n$ *and* $i = 1, \dots, k$,

$$a^{\beta}_{k,i} = \begin{cases} \frac{1}{n^{\beta}\beta(\beta+1)} \left(\frac{(\Lambda_{k,i+1})^{\beta+1} - (\Lambda_{k,i})^{\beta+1}}{\Lambda_{i+1,i}} + \frac{(\Lambda_{k,i-1})^{\beta+1} - (\Lambda_{k,i})^{\beta+1}}{\Lambda_{i,i-1}} \right), & \textit{if } 1 \leq i \leq k-1, \\ \frac{1}{n^{\beta}\beta(\beta+1)}, & \textit{if } i = k. \end{cases}$$

The Euclidean distance $\|\mathbf{F}^0[\mathcal{I}^{\beta}_{0+}y] - \underline{L}^{\beta}\mathbf{Y}\| < \varepsilon$.

Remark 3 *It is easy to see that the matrix* $\underline{L}^{\beta}$ *is not singular.*

Corollary 4 *Let the conditions of the theorem be satisfied. Then the equation (6) can be approximately rewritten as:*

$$\mathbf{Y} = \underline{L}^{\beta}\mathbf{Y} + \mathbf{G},$$

so that the direct F^0*-transform* $\mathbf{Y}$ *of the exact solution* y *of equation (1) can be found using*

$$\mathbf{Y} = (E - \underline{L}^{\beta})^{-1}\mathbf{G}.$$

The approximate solution of equation (1) can be represented as the inverse F^0*-transform, that is:*

$$\hat{y}^0_F(t) = \sum_{k=1}^{n} Y^0_k A_k(t).$$

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