

Relation of Complete Correlation and Its Implication on Interval Operations

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Abstract

This paper lays the groundwork for defining division and multiplication on fuzzy intervals under complete correlation. We introduce joint possibility distributions to model dependencies between fuzzy variables with complete correlation expressed via a linear relation. We then examine its impact on interval operations and the inverse property. Our results show that fuzzy arithmetic requires more nuanced approaches than simple point-wise interval analogies.

1 Preliminaries

We begin by introducing fuzzy intervals (or fuzzy numbers) and their α -cuts, which generalize real intervals and are essential for later definitions. A fuzzy set A on \mathbb{R} is called a **fuzzy interval** if it is normal, fuzzy convex, upper semi-continuous, and has bounded support [1]. Its **α -cuts** are defined as $[A]_\alpha = \{x \in \mathbb{R} \mid A(x) \geq \alpha\}$ for $\alpha \in (0, 1]$, and each is a closed interval $[a_\alpha^-, a_\alpha^+]$. The set of all fuzzy numbers is denoted by $\mathbb{R}_\mathcal{F}$. A fuzzy interval is **symmetric** about $x \in \mathbb{R}$ if $A(x - y) = A(x + y)$ for all y ; otherwise, it is **non-symmetric**. Next, we introduce complete correlation via joint possibility distributions and the sup-J extension principle, following [2, 3].

Definition 1 An ***n-dimensional possibility distribution*** on \mathbb{R}^n is a fuzzy subset J with a membership function $J: \mathbb{R}^n \rightarrow [0, 1]$ satisfying $J(x_0) = 1$ for some $x_0 \in \mathbb{R}^n$.

The family of n-dimensional possibility distributions will be denoted $\mathcal{F}(\mathbb{R}^n)$.

Definition 2 Let $A_1, \dots, A_n \in \mathbb{R}_\mathcal{F}$ be fuzzy intervals, then $J \in \mathcal{F}(\mathbb{R}^n)$ is called a **joint possibility distribution** if for each $i = 1, \dots, n$, the following condition holds:

$$\bigvee_{x_j \in \mathbb{R}, j \neq i} J(x_1, \dots, x_n) = A_i(x_i), \quad \forall x_i \in \mathbb{R}.$$

Each A_i is called the *i-th marginal possibility distribution* of J .

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Definition 3 Fuzzy intervals A, B are said to be **completely correlated** if there exist $q, r \in \mathbb{R}, q \neq 0$ such that their joint possibility distribution J_c is given by

$$J_c(x, y) = A(x)\chi_{\{qx+r=y\}}(x, y) = B(y)\chi_{\{qx+r=y\}}(x, y), \quad (1)$$

where $\chi_{\{qx+r=y\}}$ stands for the characteristic function of the line $\{(x, y) \in \mathbb{R} \mid qx + r = y\}$ and $[A]_\alpha = [a_\alpha^-, a_\alpha^+], [B]_\alpha = q[A]_\alpha + r, \alpha \in [0, 1]$.

Remark 4 Fuzzy intervals A and B are said to be completely **positively** (**negatively**) correlated if q is positive (negative).

Remark 5 Symmetric fuzzy intervals can be both positively and negatively correlated at the same time.

Definition 6 Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function and let $J \in \mathcal{F}(\mathbb{R}^2)$ be a joint possibility distribution of $A, B \in \mathbb{R}_{\mathcal{F}}$. The **sup-J extension** of f to a fuzzy function $f_J: \mathcal{F}(\mathbb{R}^2) \rightarrow \mathcal{F}(\mathbb{R})$ is defined by

$$f_J(A, B)(y) = \bigvee_{y=f(x,y)} J(x, y).$$

Theorem 7 Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function and let $J \in \mathcal{F}(\mathbb{R}^2)$ be a joint possibility distribution of $A, B \in \mathbb{R}_{\mathcal{F}}$. Then $[f_J(A, B)]_\alpha = f([J]_\alpha), \forall \alpha \in [0, 1]$. If $f_J(A, B)$ is a fuzzy number, then

$$[f_J]_\alpha = \left[\bigwedge_{(x,y) \in [J]_\alpha} f(x, y), \bigvee_{(x,y) \in [J]_\alpha} f(x, y) \right].$$

2 Relation between Division and Multiplication of Completely Correlated Intervals

In this section, we define the operations division and multiplication of two real intervals which are connected by the complete correlation given by Definition 3. Recall that real intervals can be interpreted as special cases of α -cuts of fuzzy intervals. In addition, we present the invertibility property between these two operations.

Definition 8 [4] Let $A = [a^-, a^+], B = [b^-, b^+], 0 \notin A$, be real compact intervals and let $J_c \in \mathcal{F}(\mathbb{R}^2)$ be the complete correlation relation of A, B defined by (1). The J_c -**division** is the real interval $B \div_{J_c} A$ defined by

$$B \div_{J_c} A = \begin{cases} [q + \frac{r}{a^-}, q + \frac{r}{a^+}], & r < 0, \\ [q + \frac{r}{a^+}, q + \frac{r}{a^-}], & r \geq 0. \end{cases} \quad (2)$$

Definition 9 Let $A = [a^-, a^+], B = [b^-, b^+]$ be real intervals, and let $J_c \in \mathcal{F}(\mathbb{R}^2)$ be the complete correlation relation of A, B defined by (1). The J_c -**multiplication** of A, B is the real interval $A \odot_{J_c} B$ defined by

$$A \odot_{J_c} B = [\min S, \max S]^* \quad \text{where} \quad S = \{a^-(qa^- + r), a^+(qa^+ + r)\}, \quad (3)$$

and $*$ defines the following exceptions:

1. $A > 0, B > 0, q < 0$ with $r \in (-qa^+, -2qa^+)$ or $A < 0, B < 0, q < 0$ with $r \in (-2qa^-, -qa^-)$:

$$A \odot_{J_c} B = \left[\min S, \bigvee_{a \in A} a(aq + r) \right].$$

2. $A < 0, B > 0, q > 0$ with $r \in (-qa^-, -2qa^-)$ or $A > 0, 0 \in B, q > 0$ with $r \in [-qa^+, -2qa^-]$

$$A \odot_{J_c} B = \left[\bigwedge_{a \in A} a(aq + r), \max S \right].$$

3. Analogous exceptions for cases $A > 0, 0 \in B$ or $A < 0, 0 \in B$.

Remark 10 Based on Remark 5, real intervals are positively and negatively correlated with respect to the parameter q . Therefore, the results of division or multiplication of such correlated intervals differ according to the particular correlation and parameter q .

Example 11 Consider the intervals $A = [3, 6]$ and $B = [6, 15]$.

1. A, B are positively correlated with $q = 3, r = -3$.
Then $B \div_{J_c} A = [2, \frac{5}{2}]$ and $A \odot_{J_c} B = [18, 90]$.
2. A, B are negatively correlated with $q = -3, r = 24$.
Then $B \div_{J_c} A = [1, 5]$ and $A \odot_{J_c} B = [36, \bigvee_{a \in A} a(-3a + 24)]$.

Proposition 12 For the following combinations:

1. $q > 0$ and $A > 0, B > 0$ or $A < 0, B < 0$,
2. $q < 0$ and $A > 0, B < 0$ or $A < 0, B > 0$,

the bounds of J_c -multiplication coincide with the standard interval multiplication.

Theorem 13 Let $A = [a^-, a^+], B = [b^-, b^+]$ be real intervals, and let $J_c \in \mathcal{F}(\mathbb{R}^2)$ be the complete correlation relation of A, B defined by (1). Let $C = [c^-, c^+]$ be the real interval such that $C = B \div_{J_c} A$ is given by (2). Then there exists $J_{c'} \in \mathcal{F}(\mathbb{R}^2)$ of A, C given by (1) with parameters q' and r' such that the following holds

$$C = B \div_{J_c} A \Rightarrow B = A \odot_{J_{c'}} C.$$

Proof Assume $A = [a^-, a^+], B = qA + r = [\min\{qa^- + r, qa^+ + r\}, \max\{qa^- + r, qa^+ + r\}]$, and $C = B \div_{J_c} A$. By the definition of J_c -division, $C = [\min\{q + \frac{r}{a^-}, q + \frac{r}{a^+}\}, \max\{q + \frac{r}{a^-}, q + \frac{r}{a^+}\}]$. Based on the structures of A, B and C , we can always define a complete correlation $J_{c'}$ between A and C such that $A \odot_{J_{c'}} C = B$. ■

Theorem 14 Let $A = [a^-, a^+], B = [b^-, b^+]$ be real intervals, and let $J_c \in \mathcal{F}(\mathbb{R}^2)$ be the complete correlation relation of A, B defined by (1). Let $C = [c^-, c^+]$ be the real interval such that $C = B \div_{J_c} A$ is given by (2). Then there exists $J_{c'} \in \mathcal{F}(\mathbb{R}^2)$ of A, C or B, C^{-1} given by (1) with parameters q' and r' such that the following equivalence holds

$$C = B \div_{J_c} A \iff \begin{cases} (i) & B = A \odot_{J_{c'}} C, \\ (ii) & A = B \odot_{J_{c'}} C^{-1}, \quad \text{where } C^{-1} = [\frac{1}{c^+}, \frac{1}{c^-}]. \end{cases} \quad (4)$$

Proof The complete proof is extensive, but we focus here on the principal concepts employed to demonstrate this theorem. In the forward direction, we rely on the proof of Theorem 13 and the concept of g -division, which is reversible using standard interval multiplication. Conversely, we apply Proposition 12 along with the notion that one of the multiplications in (4) consistently results in the precise boundaries specified by (3). ■

2.1 Extension to Fuzzy Intervals

In this subsection, we present a remark that links the J_c -arithmetic of real intervals with fuzzy intervals and briefly formalize this extension.

Remark 15 *The invertibility result for intervals does not generally extend to fuzzy intervals. Let $A, B \in \mathbb{R}_{\mathcal{F}}$ with α -cuts $[A]_{\alpha} = [a_{\alpha}^-, a_{\alpha}^+]$ and $[B]_{\alpha} = q[A]_{\alpha} + r$ be fuzzy intervals under the completely correlated joint possibility distribution J_c as defined in (1). Applying the invertibility theorem levelwise for each $\alpha \in [0, 1]$,*

$$[C]_{\alpha} = [B \div_{J_c} A]_{\alpha}, \quad [B]_{\alpha} = [A \odot_{J_c} C]_{\alpha}, \quad [A]_{\alpha} = [B \odot_{J_c} C^{-1}]_{\alpha},$$

may yield intervals $\{[C]_{\alpha}\}$ that violate fuzzy interval properties, such as convexity and monotonicity of endpoints.

3 Conclusion

In this contribution, we present J_c -division and J_c -multiplication for real intervals. These operations are defined by the particular joint possibility distribution J_c called complete correlation. Moreover, we investigate the inverse property between the defined operations. Although the structure of J_c -arithmetic for intervals is elegant and invertible, its naive extension to fuzzy numbers through α -cuts introduces inconsistencies. These findings indicate that the theory of interactive fuzzy arithmetic demands more refined constructions beyond point-wise interval analogies.

References

- [1] Ahmad, M.Z., De Baets, B.: *A Predator-Prey Model with Fuzzy Initial Populations*. The Joint 2009 International Fuzzy Systems Association World Congress and 2009 European Society of Fuzzy Logic and Technology (2009) 1311–1314.
- [2] Cabral, V.M., Barros, L.C.: *Fuzzy differential equation with completely correlated parameters*. Fuzzy Sets and Systems. **265** (2015) 86–98.
- [3] Carlsson, C., Fuller, R., et al.: *Additions of completely correlated fuzzy numbers*. 2004 IEEE International Conference on Fuzzy Systems (IEEE Cat. No. 04CH37542). **vol. 1** (2004) 535–539.
- [4] Alijani, Z., Števuliáková, P.: *Division of interactive fuzzy numbers using joint possibility distributions*. Information Processing and Management of Uncertainty in Knowledge-Based Systems. IPMU 2024. Lecture Notes in Networks and Systems. Springer, Cham. **vol. 1175** (2025) 271–282.