Verification of Validity of Syllogisms Related to Graded Peterson Cube of Opposition

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Abstract

In this article, we will examine the validity of selected forms of logical syllogisms with intermediate quantifiers. We will focus in particular on forms related to the graded Peterson's cube of opposition. Our verification will be based on the application of graded Peterson's rules using the distribution index.

Keywords: Fuzzy natural logic, Intermediate quantifiers, Distribution index, Graded cube of opposition

1 Introduction

This paper is a contribution to methods of verification of the validity of logical syllogisms with intermediate quantifiers. In previous papers, we focused mainly on formal verification of validity, i.e., we constructed a syntactic proof (see [1, 2]). Now we turn our attention to verification using generalized Peterson rules introduced in [3]. Since they are formulated in natural language, we proposed their mathematical formulation in [4]. In [5] we proved that a syllogism is valid iff all Peterson rules are satisfied. In [6] we introduced modification of these rules to syllogisms related to the graded Peterson's cube of opposition. The following are examples quantifiers from the latter.

- Most people who do not exercise regularly are at higher risk of heart disease.
- Almost all individuals who do not eat breakfast report feeling tired before noon.
- A few people who do not drink enough water suffer from reduced concentration.

Recall that many authors addressed other selected forms of logical syllogisms. Generalized syllogisms with interval quantifiers were developed in [7]. In connection with this, also extended syllogistic reasoning with the new quantifiers was proposed (see for example, [8, 9]).

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2 Preliminaries

2.1 Mathematical bacground

Intermediate quantifiers are formally defined within a formal theory T^{IQ} of (Łukasiewicz) Fuzzy Type Theory (Ł-FTT). The underlying algebra of truth values is an MV$_\Delta$-algebra $\mathscr{E} = \langle E, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}, \Delta\rangle$ where $\mathbf{0}$ and $\mathbf{1}$ are the minimal and maximal elements, respectively. In most cases, we assume that the model is the standard Łukasiewicz MV$_\Delta$-algebra whose domain of truth degrees is $E = [0, 1]$. Each formula of Ł-FTT is assigned a type $\alpha \in$ *Types* representing a certain kind of elements.

2.2 Mathematical definition of intermediate quantifiers

Definition 1 (Fuzzy intermediate quantifiers relate to cube) *Let $Ev \in Form_{oo}$ be a formula representing an evaluative expression, x_α is a variable and $A, B, z \in Form_{o\alpha}$ be formulas representing fuzzy sets. Then either of the formulas*

$$(Q^{\forall}_{Ev}\, x)(\neg B, \neg A) \equiv (\exists z)[(\forall x)((\neg B|z)\, x \Rightarrow \neg Ax) \wedge Ev((\mu(\neg B))(\neg B|z))], \tag{1}$$

$$(Q^{\exists}_{Ev}\, x)(\neg B, \neg A) \equiv (\exists z)[(\exists x)((\neg B|z)x \wedge \neg Ax) \wedge Ev((\mu(\neg B))(\neg B|z))]. \tag{2}$$

construes the sentence "⟨Quantifier⟩ not B's are not A".

For instance, the quantifier "Most" is determined by the evaluative expression *Bi Ve*, i.e., *very big*. For the details see [10].

Lemma 2 ([6]) *The following system of inequalities holds for extensions of evaluative expressions*[†)] *in the standard context:*

$$0 < \inf \mathrm{Supp}(\neg\, Sm\, Si) \leq \inf \mathrm{Supp}(\neg\, Sm\ Ve) \leq a_{\neg\, Sm\, \bar{\boldsymbol{\nu}}} = \inf \mathrm{Supp}(\neg\, Sm\, \bar{\boldsymbol{\nu}}) < 0.5$$
$$< a_{Bi\, Ve} = \inf \mathrm{Supp}(Bi\, Ve) \leq a_{Bi\, Ex} = \inf \mathrm{Supp}(Bi\, Ex) \leq \inf \mathrm{Supp}(Bi\, \boldsymbol{\Delta}) = 1.$$

3 Verification of selected forms of logical syllogisms

3.1 Peterson's rules

Extended Peterson's rules for verification of the validity of logical syllogisms related to the graded cube of opposition were initially proposed in [6]. Essential for them is the concept of *distribution index* (see [4, 5]).

Definition 3 (Distribution index-second face) *Let N be a set, $A, B \in \mathscr{F}(N)$.*
Let $Q^K_{Ev}(\neg B, \neg A)$, $K \in \{\forall, \exists\}$ be an intermediate quantifier determined by the evaluative expression Ev. The distribution index $\mathrm{DI}(X, Q^K_{Ev}(\neg B, \neg A))$ *(or shortly,* $\mathrm{DI}(X, Q^K)$*) of the fuzzy set $X \in \mathscr{F}(N)$ is:*

$$\mathrm{DI}(X, Q^K_{Ev}(\neg B, \neg A)) = \begin{cases} \inf \mathrm{Supp}(Ev) & \text{if } K = \forall \text{ and } X = \neg B, \\ 0 & \text{if } K = \forall \text{ and } X = \neg A, \\ 0 & \text{if } K = \exists \text{ and } X \in \{\neg A, \neg B\}, \end{cases}$$

$$\mathrm{DI}(X, Q^K_{Ev}) = \neg\, \mathrm{DI}(\neg X, Q^K_{Ev}).$$

[†)]By $\neg\, Sm$ and etc. we denote a group of evaluation expressions, e.g. not small, significantly small, very big, extremely big, which are constructed with the basic trichotomy small, medium, big and modified by a specific hedge. *Support* of A is a set $\mathrm{Supp}(A) = \{u \in N \mid A(u) > 0\}$. For details see Figure 1 in [11].

Formalization of Peterson's rules

1. *Rules of Distribution*

   Let $K, L \in \{\forall, \exists\}$.

(fER1) $\mathrm{DI}(X, Q^K_{\mathscr{P}_1} \oplus \mathrm{DI}(X, Q^L_{\mathscr{P}_2} = 1$ where $X \in \{M, \neg M\}$.

(fER2a) $\mathrm{DI}(X, Q^K_{\mathscr{C}}) \leq \mathrm{DI}(Y, Q^L_{\mathscr{P}_2})$ where $X, Y \in \{S, \neg S\}$.

(fER2b) $\mathrm{DI}(X, Q^K_{\mathscr{C}}) \leq \mathrm{DI}(Y, Q^L_{\mathscr{P}_1})$ where $X, Y \in \{P, \neg P\}$.

2. *Rules of Quality second face*

(fER3) Let $X, Y \in \{S, P, M\}$, $X \neq Y$ and $K \in \{\forall, \exists\}$. Then at least one of the following must hold: $\mathscr{P}_1 = Q^K_{Ev}(\neg Y, \neg X)$ or $\mathscr{P}_2 = Q^K_{Ev}(\neg Y, \neg X)$.

(fER4) Let $X, Y \in \{S, P, M\}$, $X \neq Y$ and $K, L \in \{\forall, \exists\}$. Then

$$\mathscr{C} = Q^K_{Ev}(\neg S, P) \quad \text{iff} \quad \mathscr{P} = Q^L_{Ev}(\neg Y, X)$$

where $\mathscr{P} \in \{\mathscr{P}_1, \mathscr{P}_2\}$.

3.2 Verification of validity of logical syllogisms

In this subsection, we will demonstrate application of extended Peterson's rules to validity of a logical syllogism related to the second (back) face of the graded cube of opposition. We selected a non-trivial syllogism and explain in the discussion that similar syllogisms are valid with a particular conclusion only. The following is a syllogism **pti**-III:

$\mathscr{P}_1$: Almost all $\neg M$ are $\neg P$.
$\mathscr{P}_2$: Most $\neg M$ are $\neg S$.

$\mathscr{C}$: Some $\neg S$ are $\neg P$.

Example of this syllogism from the field of wellbeing is

$\mathscr{P}_1$: Almost all people who do not get regular social support do not feel emotionally connected.
$\mathscr{P}_2$: Most people who do not get regular social support do not maintain healthy daily routines.

$\mathscr{C}$: Some people who do not maintain healthy daily routines do not feel emotionally connected.

This syllogism satisfies all the extended Peterson's rules. Indeed, from Definition 3 we know that $\mathrm{DI}(\neg M, Q^{\forall}_{Bi\,Ex}(\neg M, \neg P)) = \inf \mathrm{Supp}(Bi\ Ex_{\mathscr{P}_1})$ and $\mathrm{DI}(\neg M, Q^{\forall}_{Bi\,Ve}(\neg M, \neg S)) = \inf \mathrm{Supp}(Bi\ Ve_{\mathscr{P}_2})$. Applying of Lemma 2 we conclude that Rule (fER1) is satisfied. Furthermore, we know that $\mathrm{DI}(\neg S, Q^{\forall}_{Bi\,Ve}(\neg M, \neg S) = 0$ as well as $\mathrm{DI}(\neg S, Q^{\exists}_{Bi\,\boldsymbol{\Delta}}(\neg S, \neg P)) = 0$. It means that the Rule (fER2a) is fulfilled. Finally, $\mathrm{DI}(\neg P, Q^{\forall}_{Bi\,Ex}(\neg M, \neg P) = 0$ as well as $\mathrm{DI}(\neg P, Q^{\exists}_{Bi\,\boldsymbol{\Delta}}(\neg S, \neg P)) = 0$. It means that the Rule (fER2b) is fulfilled. Rules (fER3) and (fER4) are trivially fulfilled.

Now let us consider this syllogism with the quantifier "A few" in the conclusion. Then it does not satisfy Rule (fER2b). Indeed, using Definition 3 we obtain $\mathrm{DI}(\neg P, Q^{\forall}_{Bi\,Ex}(\neg M, \neg P)) = 0$. Furthermore, $\mathrm{DI}(\neg P, Q^{\forall}_{\neg\,Sm\,Ve}(\neg S, \neg P)) = \inf \mathrm{Supp}(\neg\ Sm\ Ve_{\mathscr{C}}) > 0$ and so, Rule (fER2b) is not satisfied.

4 Conclusion

In this paper, we have focused on further developing extended Peterson rules related to verifying the validity of logical syllogisms generated from the second face graded Peterson cube of opposition. We verified the proposed rules on a valid logical syllogism of the third figure, which is related to the area of wellbeing. In future publications, we will focus on another study that will be devoted to other selected forms of logical syllogisms.

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