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FSTA 2026

Edited by

Andrea Stupňanová, Martin Dyba, Viktor Pavliska

UNIVERSITY OF OSTRAVA
INSTITUTE FOR RESEARCH AND APPLICATIONS OF FUZZY MODELING

STU
SLOVAK UNIVERSITY OF TECHNOLOGY IN BRATISLAVA

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International Conference on Fuzzy Set Theory and Applications FSTA 2026

BACZYŃSKI Michał

Faculty of Science and Technology, University of Silesia in Katowice
Katowice, Poland

Title: **New Horizons in Fuzzy Logic Connectives: Structural Theory and Recent Advances**

This plenary lecture surveys selected developments from recent years in the theory of fuzzy logic connectives, with emphasis on conjunctions (t-norms), disjunctions (t-conorms), and fuzzy implication functions. The focus is on algebraic and order-theoretic properties, construction principles, representation issues, and comparative frameworks for different classes of connectives. Joint works of Baczyński-Dombi, Baczyński-Reformat, and Baczyński-Kaczmarek-Miś are discussed. Selected applications of fuzzy logic connectives in machine learning are also outlined. The talk concludes with an outlook on future research directions.

BUSTINCE Humberto

Universidad Pública de Navarra
Pamplona, Spain

Title: **From Choquet-inspired to Choquet-functional integrals**

In recent years, it is developing a growing interest on the use of fuzzy integrals [3] for information fusion in different processes, ranging from deep learning to classification [2]. This is due to the abaility of these techniques to represent relationships between data by means of an appropriate choice of the involved fuzzy measures. However, one of the main drawbacks of fuzzy integrals lays precisely on the difficulty and the computational cost of building the measures, as well as in the need to make a previous ordering of the inputs.

For these reasons, some researchers, including my group, have been devloping new functions to fuse information which follow the spirit of Choquet integrals (as one of the main representatives of fuzzy integrals) but may provide a way to overcome this problems. In this talk, we are going to discuss the notion of Choquet-inspired function [1], as a function which mimics the structure of a discrete Choquet integral but replaces the measure by other functions, hence providing a more general framework to work. We will also show how these Choquet-inspired functions can be further extended to provide the so-called Choquet-functional integrals, which can be seen as a very general tool with many different applications. We will present all the relevant definitions, the main properties and the relation with usual fuzzy integrals. We will discuss some applications where these new approaches have shown themselves useful.

Acknowledgements: This work has been supported by research project PID2022-136627NB-I00, Spain (MCIN/AEI/10.13039/501100011033/ FEDER, UE)

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DE BAETS Bernard

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Ghent University, Belgium
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Title: **Law & Order: New Kids on the Block**

Shortly after Lotfi Zadeh introduced fuzzy sets in the 1960s, Joe Goguen recognized that a lattice-theoretic framework — particularly that of complete residuated lattices — provides the most suitable foundation for developing fuzzy set theory. This challenge was taken up by numerous researchers, contributing not only to fuzzy set theory itself but also to the advancement of lattice theory. A similar observation applies to the fields of formal concept analysis and mathematical morphology.

While the lattice-theoretic approach centers on the notion of an order relation (combining reflexivity, antisymmetry, and transitivity), an equally important role in fuzzy set theory is played by equivalence relations (combining reflexivity, symmetry, and transitivity) and by their transitive and non-transitive fuzzy generalizations. Notably, the transitive notions of order and equivalence converge in the context of pre-order relations.

Given the close connection between fuzzy set theory and decision-making, even more general relational structures have been explored. Of particular importance are (fuzzy) preference structures consisting of a strict preference (generalizing strict order relations), an indifference (generalizing equivalence relations), and an incomparability component. The property of transitivity in this setting requires careful reflection and analysis. A notable example is that of pseudo-order relations, which may be non-transitive and can accommodate cycles.

A characteristic feature of lattices is the existence of meet and join operations, binary operations that fulfill the laws of monotonicity and associativity. The concepts of lattices and pseudo-order relations jointly inspired Helen Skala in the 1970s to introduce trellises, a generalization of lattices that preserves the existence of meet and join operations while abandoning the transitivity of the underlying pseudo-order relation. Similarly, sponges represent a more recent generalization of complete lattices. Trellises have only recently attracted the attention of the fuzzy community.

In this lecture, we provide a comprehensive overview of these structures, discussing both the motivations for and the caveats against their study in fuzzy set theory. In particular, we focus on alternatives to the monotonicity property, since in proper trellises the meet and join operations are no longer monotone (nor associative) and we explore the notion of triangular norms in this context.

FLAMINIO Tommaso

Artificial Intelligence Research Institute IIIA - CSIC, 08193
Bellaterra, Spain

Title: **Fuzzy Logic and Probability Theory are Cooperative (Rather than Complementary or Competitive)**

Since its inception, fuzzy logic has been frequently compared with (and sometimes equated to) probability theory. Over the years, several attempts have been made to clarify the philosophical, mathematical, logical, and applicative differences between these two theories. Interestingly, two crucial papers addressing this topic were published in 1995:

(1) "Probability Theory and Fuzzy Logic Are Complementary Rather Than Competitiv", by Zadeh, and

(2) "Probability and Fuzzy Logic", by Hájek, Esteva, and Godo.

The present talk aims to draw on recent literature that builds upon the paper by Hájek, Esteva, and Godo (2) in order to offer an additional perspective on the discussion initiated by Zadeh's paper (1). Our goal, therefore, is not to further clarify the distinction between fuzzy logic and probability (or uncertainty in general). Rather, we aim to support the idea that these two theories often operate in a cooperative manner.

We will support our general claim by showing how fuzzy logic aids probability theory in its generalization, and how probability theory can be (partially) reduced, interpreted, and represented within a formal fuzzy-logical framework.

RICO Agnès

Claude Bernard University Lyon 1
Lyon, France

Title: **Integral Explanation**

Today, decision support systems are widely used to help humans make decisions when selecting objects or alternatives. Qualitative representations and their use in reasoning processes have long been part of artificial intelligence (AI) since they are well aligned with human cognition and reasoning. One of their main advantages is their naturalness, as well as their ability to support reasoning with limited data. Although qualitative approaches are less expressive than quantitative ones, they can provide more robust results with much less effort. In this context it is natural to look for qualitative aggregation operators.

Sugeno integrals are commonly used as aggregation functions in decision theory within a symbolic framework. They compute a global evaluation of alternatives or objects assessed on several criteria using an ordinal evaluation scale. They are based on set functions called capacities, or fuzzy measures. Generalized versions of Sugeno integrals extend the way the capacity value of each subset of criteria is combined with the utility values of the elements in the subset. This generalized notion of the Sugeno integral can be split into two functionals. In this talk, we focus on a particular case: Gödel integrals. Moreover both Sugeno integrals and Gödel integrals may be elicitated to represent and explain a dataset.

In conclusion, Gödel integrals are applied in the field of explainable AI (XAI) to provide explanations that are better adapted to users, particularly in the case of counterfactual examples. An interactive incremental algorithm to elicit capacities for Gödel integrals, applied to the aggregation of two types of criteria present in counterfactual examples selection is presented.

VOMLEL Jiří

ÚTIA, Czech Academy of Sciences, Prague
and
IRAFM, University of Ostrava, Ostrava
Czech Republic

Title: **Fuzzy Transforms Meet Bayesian Networks in Sociological Data Analysis**

This talk explores the synergy between fuzzy set theory and Bayesian networks for modeling uncertainty in real-world applications. We demonstrate how the F-transform, a fundamental tool in fuzzy approximation theory, can be leveraged to achieve computational efficiency in probabilistic inference while maintaining interpretability–particularly when dealing with ordinal data from surveys and questionnaires.

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Proceedings Papers

International Conference on Fuzzy Set Theory and Applications FSTA 2026

Iterated Correlation Under Uncertainty: A Three-Layer Predictive Model with Fuzzy Upper Bounds

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The early applications of iterated correlations for clustering and block-modeling go back to psychometrics and social networks [4, 6], with later visualization work through successive correlation sharpening [5]. We study the discrete dynamical system obtained by iterating the correlation operator. Starting from a real matrix $P_0 \in \mathbb{R}^{n\times n}$ with entries drawn i.i.d. from $U[-1,1]$, one step applies Pearson row–row correlation across the n columns to form

$$\big[P_{k+1}\big]_{ij} = \frac{\sum_{\ell=1}^{n}\big((P_k)_{i\ell}-\bar{p}_i\big)\big((P_k)_{j\ell}-\bar{p}_j\big)}{\sqrt{\sum_{\ell=1}^{n}\big((P_k)_{i\ell}-\bar{p}_i\big)^2}\,\sqrt{\sum_{\ell=1}^{n}\big((P_k)_{j\ell}-\bar{p}_j\big)^2}},$$

where $\bar{p}_i$ is the mean of the row i. Computation is repeated until element-wise entries stabilize with tolerance 10^{-12}:

$$\max_{i,j}\big|(P_{k+1})_{ij}-(P_k)_{ij}\big| \le 10^{-12}.$$

We ran 10^3 trials per size n in the set

$$\{3,6,7,9,12,16,23,30,40,50,55,69,80,100,150,250,350,600,1054,1600,1700,2000\},$$

and for each run we recorded the trajectory and the stepwise quantities

$$\Delta_k := \|P_{k+1}-P_k\|_F, \qquad \rho_k := \frac{\Delta_{k+1}}{\Delta_k}.$$

Goal. This study characterizes the dynamics of the iterated correlation, identifying four empirical laws that remain stable across matrix sizes and random initializations. The sequence (P_k) exhibits a dimension-independent contraction geometry that enables systematic stepwise prediction. Building on this structure, we develop a three-layer uncertainty-aware framework that remains reliable under sampling noise, compounding uncertainty, and decision imprecision.

1. **Universal first-step contraction:** The initial ratio $\rho_0 = \Delta_1/\Delta_0$ is uniformly small ($\rho_0 \ll 1$), producing a sharp, dimension-independent early drop in Δ_k.

2. **Nearly monotone convergence:** For $k \ge 1$, Δ_k decays almost monotonically; $\rho_k \approx 1$ with bounded, vanishing oscillations; occasional overshoots ($\rho_k > 1$) are uniformly bounded.

3. **Uniformly bounded iteration counts:** The number of steps to reach a fixed tolerance ($\Delta_k < 10^{-12}$) remains in a tight band across all tested n, suggesting $T(n,\varepsilon) = O(1)$ empirically.

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4. **Universal V-shape, dimension-independent contraction law:** Pooling (Δ_k, ρ_k) on log–log axes yields a single V-shaped curve: strong contraction for large Δ_k, a valley near $\Delta^\star$ ($\rho(\Delta^\star) \approx 10^{-2}$–$10^{-1}$), then a rise toward 1 as $\Delta_k \to 0$, with bounded right-branch overshoots resembling self-normalizing maps such as matrix scaling [7] and iterative normalization in deep networks [8], though the present operator is neither linear nor globally contractive.

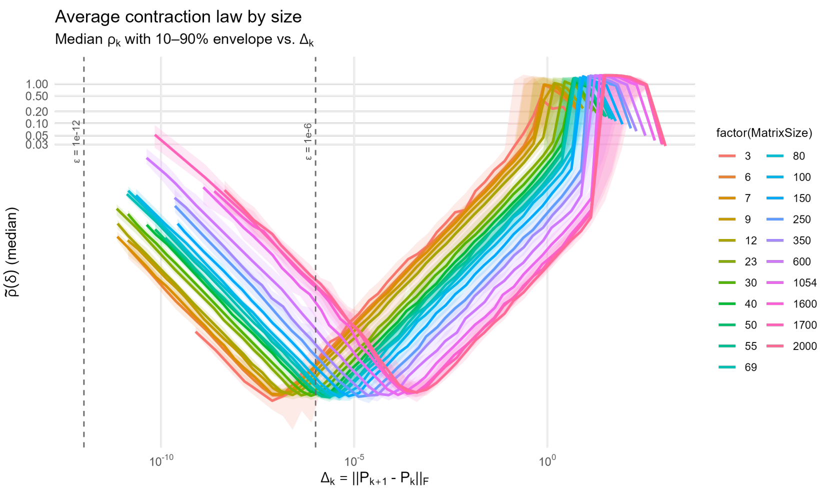

Figure 1: V-shape: median ribbons of ρ versus raw step size Δ (log–log) across matrix sizes n. The location of the valley $\Delta^\star$ and the bounded right-branch overshoot are consistent across sizes.

Geometric rationale and ongoing theoretical work. The four empirical laws arise from the geometry of the composite map $f = \Phi \circ \Psi$ defining the iteration $P_{k+1} = f(P_k)$, where Ψ performs row-centering, projecting onto the mean-zero hyperplane, and Φ applies normalization followed by a Gram lift. At large steps, Ψ acts as a strong projection, producing the sharp contraction that forms the left branch of the V-shape. Near equilibrium, Φ and Ψ nearly cancel along tangent directions, yielding almost distance-preserving updates (right branch). For the sequence $\{P_k\}_{k\geq 0}$, the cumulative total variation up to step K is

$$V_K := \sum_{k=0}^{K-1} \Delta_k,$$

which empirically converges to a finite limit $V_\infty \leq C$ with C independent of n. Since $V_\infty < \infty$ implies $\Delta_k \to 0$, the flow $P_{k+1} = f(P_k)$ stabilizes in $O(1)$ steps, uniformly in n. A theoretical study via global descent and local geometric analysis is underway.

From geometry to predictive bounds. To translate this contraction geometry into a predictive framework, we first extract an empirical median response $g(\Delta)$ capturing the pattern of (Δ, ρ) across sizes. This baseline curve then serves as the foundation for three predictive layers: (i) a thin-tail upper q_p^{naive} addressing sampling variability; (ii) a tail-conservative inflation q_p^{TC} controlled by τ to capture compounding "error-on-error" uncertainty in the sense of [1]; and (iii) a fuzzy bound U_α introducing decision-level tolerance. Together, these yield a family of bounds parameterized by $(p, \tau, \alpha, \lambda)$ while preserving the geometry revealed in Figure 1.

Baseline. From (Δ_k, ρ_k) we drop the first two transitions, optionally normalize by n via $\delta_k = \Delta_k/n$, trim the 1%–99% tails of Δ, and bin Δ on a logarithmic grid $\{B_j\}_{j=1}^K$ with centers Δ_j. Within each bin we work in the log-domain to stabilize multiplicative noise and then exponentiate back so that $g(\Delta)$, $\sigma(\Delta)$, and all predictive layers are expressed on the original ρ-scale:

$$m(\Delta_j) \;:=\; \operatorname{median}\bigl(\log\rho \bigm| \Delta \in B_j\bigr), \qquad q^{\log}_{\gamma}(\Delta_j) \;:=\; \operatorname{quantile}_{\gamma}\bigl(\log\rho \bigm| \Delta \in B_j\bigr).$$

The baseline median response is $g(\Delta) \;:=\; \exp\bigl(m(\Delta)\bigr)$, with compatible dispersion

$$\sigma(\Delta) \;:=\; \frac{\exp\bigl(q^{\log}_{\gamma}(\Delta)\bigr) - g(\Delta)}{z_\gamma}, \quad z_\gamma := \Phi^{-1}_{\mathcal{N}}(\gamma), \quad \gamma = 0.90,$$

where γ is a fixed quantile level used only for estimating scale and z_γ is its associated standard-normal upper quantile.

Layer 1: Thin-Tail Predictive Upper. Here $p \in (0.5, 0.999)$ denotes a target right-tail probability and $z_p := \Phi^{-1}_{\mathcal{N}}(p)$ is the corresponding standard-normal upper quantile. We define the *thin-tail* predictive upper quantile for $\rho \mid \Delta$ by

$$q^{\text{naive}}_{p}(\Delta) \;:=\; g(\Delta) \;+\; \sigma(\Delta)\, z_p.$$

This construction respects the V-shaped median geometry via $g(\Delta)$ and adjusts for sampling variability through the local scale $\sigma(\Delta)$.

Layer 2: Tail-Conservative Inflation. Even stable bin-wise fits may *understate* uncertainty in the extreme right tail because model mis-specification compounds (*error-on-error*) across transformations, as emphasized in the regress-of-uncertainty argument of [1]. We model this compounding scale uncertainty by a multiplicative factor $E \sim \text{LogN}(0, \tau^2)$, where $\tau > 0$ controls the magnitude of this effect. A standard log-domain argument shows that compounding scale error yields the inflation

$$\text{infl}(\tau) \;:=\; \exp\Bigl(\tfrac{1}{2}\,\tau^2\Bigr),$$

and therefore the *tail-conservative* predictive upper

$$q^{\text{TC}}_{p}(\Delta) \;:=\; g(\Delta) \;+\; \sigma(\Delta)\,\text{infl}(\tau)\, z_p.$$

Here τ tunes the strength of the error-on-error correction without altering the learned V-shaped geometry encoded in $g(\Delta)$.

Layer 3: Fuzzy Bound. Let $\alpha \in [0, 1]$ be an *acceptance* level (higher α = tighter bound) and $\lambda > 0$ a width parameter. We model a dilation in the sense of fuzzy-set theory [2, 3] by the α-cut of a linear membership above q^{TC}_{p}, giving the fuzzy upper:

$$U_\alpha(\Delta) \;:=\; \bigl(1 + \lambda(1 - \alpha)\bigr)\, q^{\text{TC}}_{p}(\Delta).$$

Here $U_1 = q^{\text{TC}}_{p}$ and $U_0 = (1 + \lambda)\, q^{\text{TC}}_{p}$; varying α traces a family of admissible upper bounds consistent with a chosen tolerance.

Properties. For fixed (τ, λ), the maps $\Delta \mapsto q^{\text{naive}}_{p}(\Delta)$, $\Delta \mapsto q^{\text{TC}}_{p}(\Delta)$, and $\Delta \mapsto U_\alpha(\Delta)$ all refine the baseline $g(\Delta)$ in an order-preserving manner on the ρ-scale (each lies above g and respects its V-shaped geometry). They are monotone in both the right-tail probability p and the tolerance parameter $(1 - \alpha)$, so increasing either produces a larger upper bound. They are

Replace σ by σE; at high p, for a standard normal Z one has $Q_p(\sigma E\, Z) \approx \sigma\, e^{\frac{1}{2}\tau^2}\, z_p$.

also scale-equivariant in Δ and portable across n: replacing Δ by $\delta = \Delta/n$ re-parameterizes but does not change the inherited dimension-independent shape of the median curve $g(\Delta)$.

Result. The three layers turn the universal V-shape into uncertainty-aware bounds. q_p^{naive} provides a reproducible predictive upper; q_p^{TC} adjusts for compounding error-on-error uncertainty in the spirit of [1]; and U_α introduces controllable tolerance through fuzzy dilation [2, 3]. Data determine the empirical geometry through $g(\Delta)$ and $\sigma(\Delta)$, while $(p, \tau, \alpha, \lambda)$ act as interpretable tuning parameters. This construction cleanly separates the geometry learned from the data from the user-controlled parameters that encode tolerance and caution. At representative scales, as shown in Figure 2, the three bounds preserve a strict ordering:

$$q_p^{\text{naive}}(\Delta) \;<\; q_p^{\text{TC}}(\Delta) \;<\; U_{0.9}(\Delta) \;<\; U_{0.5}(\Delta) \qquad \text{for all } p \in [0.80, 0.999],$$

confirming stability and controllable risk tolerance through p and α.

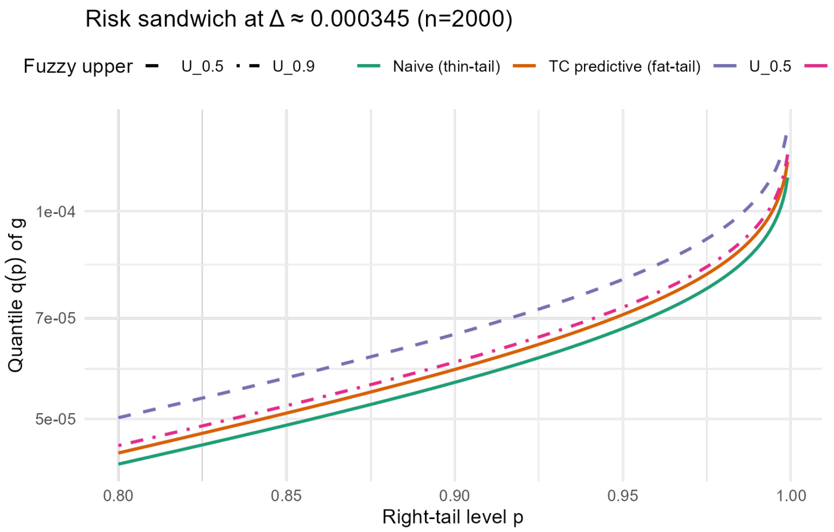

Figure 2: Layered bounds at a representative scale (75% step-size bin): thin-tail q_p^{naive}, tail-conservative q_p^{TC}, and fuzzy uppers $U_{0.5}, U_{0.9}$ traced over $p \in [0.80, 0.999]$.

Conclusion. The iterated correlation dynamics exhibit a dimension-independent V-shaped contraction arising from the finite-variation trajectory of (P_k). The three-layer framework translates this geometry into actionable predictive tools for real-world computation. Parameterized by $(p, \tau, \alpha, \lambda)$, the resulting bounds are reproducible, interpretable, and suitable for adaptive stopping, consistency checks, and control in normalization-driven or learning algorithms.

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A General Framework for Context-Aware Fuzzification of Four Ordered Categories: A Case Study on BMI Categories

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Abstract

This paper presents a general methodological framework for constructing context-aware fuzzy partitions that extend conventional crisp categorizations. The approach is based on Novák's theory of fuzzy contexts and is implemented using the R package `lfl`. It enables smooth and interpretable transitions between adjacent classes while preserving the original categorical structure. To illustrate the procedure, we apply it to derive fitness-specific fuzzy partitions of Body Mass Index, where the conventional four categories (underweight, normal weight, overweight, obese) are adapted according to individual levels of cardiorespiratory fitness.

1 Methodological Framework

We propose a **methodological, two-stage procedure** for transforming a crisp four-class categorization into a fuzzy partition using Novák's *context-based approach*, implemented in the R language package `lfl`. The framework is general and can be applied to any ordered system of four crisp categories. For illustration, we use the conventional Body Mass Index (BMI) classes (*underweight*, *normal*, *overweight*, *obese*) and demonstrate how their boundaries can be contextually adapted based on cardiorespiratory fitness (CRF) derived from $\dot{V}O_{2\,\max}$. In more detail, we propose a data-driven methodology for defining the context of the BMI linguistic terms using Novák's theory [1] of trichotomous evaluative linguistic expressions (CTX3) that are *explicitly conditioned* on cardiorespiratory fitness (CRF) expressed by $\dot{V}O_{2\,\max}$.

Novelty. To the best of our knowledge, no previous work has learned *fitness-conditioned* fuzzy BMI partitions with VO_2-based fuzzy weights merely by modifying the given context center.

To ensure comparability of the conditioning variable across individuals, we first define a normalized, dimensionless fitness index v derived from the maximal oxygen uptake $\dot{V}O_{2\,\max}$:

$$v = \frac{\dot{V}O_{2\,\max} - \dot{V}O_{2\,\max}^{\min}}{\dot{V}O_{2\,\max}^{\max} - \dot{V}O_{2\,\max}^{\min}}, \qquad v \in [0, 1],$$

where $\dot{V}O_{2\,\max}^{\min}$ and $\dot{V}O_{2\,\max}^{\max}$ denote the minimal and maximal observed (or expected) values of $\dot{V}O_{2\,\max}$ in the population. This index could later be refined to account for age and sex effects.

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- $v = 0$ corresponds to very low cardiorespiratory fitness,
- $v = 1$ corresponds to excellent cardiorespiratory fitness.

This normalization establishes a *context variable* that allows for fuzzy modification of the originally crisp category boundaries. In the second stage, each crisp boundary midpoint c is replaced by a *context-dependent center* $c'(v)$, producing a $\dot{V}O_{2\,\max}$-driven fuzzy partition of the BMI domain:

$$c'(v) = c + \alpha\,(v - 0.5),$$

where α is a sensitivity parameter that controls how strongly the boundary shifts with fitness. Typical values of α range between 4 and 6, although its calibration should rely on empirical data or expert domain knowledge.

For instance, with $\alpha = 6$ and an original threshold $c = 25$, we obtain

$$c'(0) = 22, \qquad c'(1) = 28,$$

which is consistent with epidemiological evidence indicating that individuals with high $\dot{V}O_{2\,\max}$ ("fit but overweight") exhibit health risks comparable to those of normal-weight individuals.

This example illustrates how Novák's fuzzy context modeling provides a principled way to overcome the limitations of crisp classification by allowing smooth and interpretable transitions between categories while preserving the original four-class structure. Full algorithmic details and implementation are available in the `lfl` package; due to space constraints, we summarize only the main ideas here.

2 Introduction to BMI

The classical BMI is defined as $\text{BMI} = \frac{m}{h^2}$, where m denotes body mass (kg) and h denotes height (m). The standard categorization of individuals based on the World Health Organization (WHO) BMI standards is the following:

1. **Underweight (BMI < 18.5)** High risk for nutritional deficiencies.
2. **Normal weight (BMI 18.5-24.9)** Optimal health range.
3. **Overweight (BMI 25-29.9)** Moderate risk.
4. **Obesity (BMI ≥ 30)** Severe risk of cardiovascular complications.

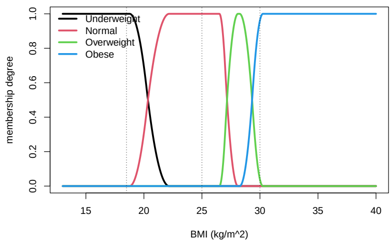
The distribution of these categories across the population provides a snapshot of overall health status.

Despite its simplicity and wide use, BMI faces many conceptual limitations that motivate ongoing research, e.g. BMI does not distinguish between fat and lean mass, optimal BMI thresholds differ across a population, health risk is not a monotonic function of BMI, and two individuals with the same BMI (e.g., athlete vs. sedentary) may have very different health statuses.

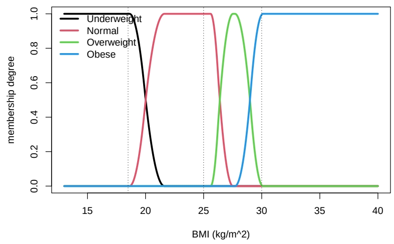
Fuzzy BMI approaches aim to overcome this limitation, but current models often rely on fixed partitions. Determining the *optimal number, shape, and overlap* of fuzzy sets that best reflect real-world health risks remains an open and critical research problem; see, e.g., fuzzy BMI indices that combine BMI with body-fat percentage for smoother classifications [2]. In more detail, fuzzy approaches replace crisp BMI thresholds with fuzzy sets and linguistic categories. In our case, we define using the preset linguistic expressions, their modifiers, and logical operations from the `lfl` package the following fuzzified classes:

under_weighted(x) = extremely_small(x),
normal(x) = not_extremely_small(x) & not_extremely_very_rougly_big(x),
over_weighted = not(normal(x)) ∨ extremely_small(x)) & not_extremely_big(x),
obese = ex_bi(x),

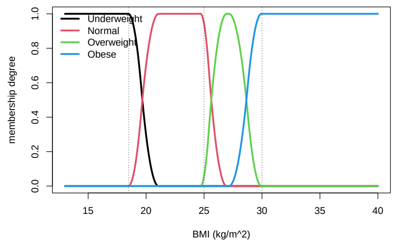
where ∨ and & stand for the Łukasiewicz disjunction and conjunction, respectively. In Figure 1, you can observe three different settings of the midpoint values of CTX3.



(a) $c = 26$



(b) $c = 25$

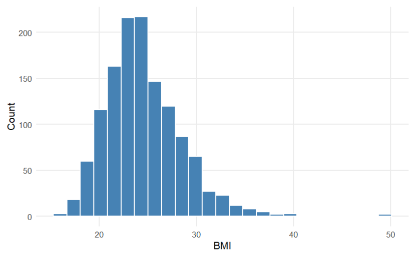


(c) $c = 24$

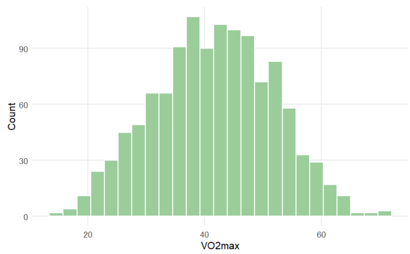
Figure 1: BMI fuzzy classes — comparison of three breakpoint settings with the fixed minimum context value 17.5 and maximum set to 31.

3 Comparison with the Standard WHO Classes

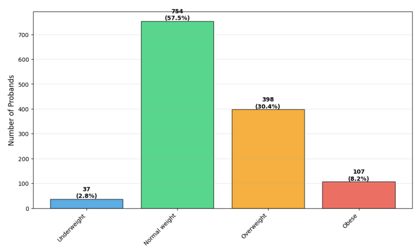
For our experiment, we use data from the Healthy Aging in Industrial Environment (4HAIE) cohort, which originally included 1,314 asymptomatic individuals. In this dataset, we observe the following distribution, see Figure 2. A Random Forest classifier is employed to classify individuals into one of four BMI categories: underweight, normal weight, overweight, and obese.



(a) Histogram of BMI



(b) Histogram of VO_2max



(c) Standard BMI classes

Figure 2: Histograms for 4HAIE data set.

Next, we used the mean decrease in impurity for model feature evaluation.

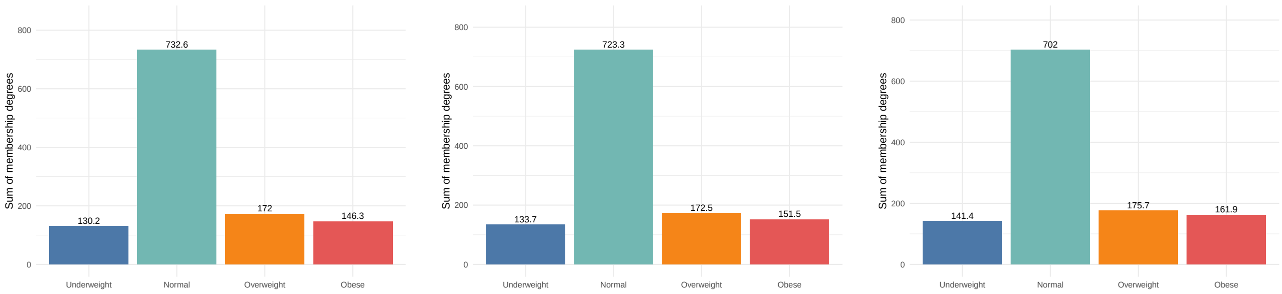
1. **Weight (36.07%)** Obviously, body weight in kilograms is the most influential feature in the model as a direct component of the BMI formula.

2. $\dot{V}O_{2\,\max}$ **(12.85%)** Maximal oxygen consumption ($\dot{V}O_{2\,\max}$) is a key indicator of cardiorespiratory fitness. Higher $\dot{V}O_{2\,\max}$ values typically correlate with better physical conditioning and lower fat mass. This feature helps distinguish between individuals with

All the data are available upon request from `https://haie-lerco.cz/data/`.

similar BMI but different fitness levels. hich are crucial for distinguishing between similar anthropometric profiles.

The remaining top five features are as follows: Age (9.85%), height (8.30%), and normalized minutes of physical activity (4.02%) significantly contribute to BMI prediction, reflecting the combined effects of age-related metabolic changes, height's moderating role in the BMI formula, and lifestyle activity patterns on body composition. Hence, we have confirmed that $\dot{V}O_{2\,\max}$ is the most influential non-defining parameter of BMI and should, therefore, be used as the principal input for individual-level BMI modification. Figure 3 illustrates three different transformations corresponding to distinct values of α.



(a) $\alpha = 4$ (b) $\alpha = 6$ (c) $\alpha = 10$

Figure 3: BMI fuzzy cardinalities for 4HAIE data with VO_2max-driven context.

4 Conclusion

The presented method is lightweight, transparent, and potentially suitable for deployment in population screening and clinical decision-support systems. It should be noted that there remains substantial room for improvement in several aspects—particularly in the choice of transformation—since $\dot{V}O_{2\,\max}$ is inherently nonlinear, and the applied transformation should adequately capture this property. Further refinement of this aspect will be the subject of future research.

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Discovering Fuzzy and Statistical Patterns in Data: The `nuggets` R Package

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The discovery of hidden logical or statistical regularities in data has long been central to data mining and knowledge discovery. In the R ecosystem, however, most available tools for this purpose—such as `arules` and `arulesViz` [7, 8]—focus narrowly on classical *association rules* or frequent itemsets, operating on crisp, categorical data and using heuristic measures such as support and confidence. An R package for subgroup discovery (e.g., `rsubgroup` [2]) or non-R rule-based learning systems (e.g., `KEEL` [1], `FURIA` [9]) broaden the scope but are respectively confined to Boolean logic and predictive modeling. The `lfl` package [4] offers the `searchrules()` function for mining *fuzzy association rules*, which partially addresses the need for fuzzy reasoning in R; however, its capabilities are limited to this single pattern type and lack the extensibility required for more general fuzzy or statistical analyses.

The `nuggets` package for R statistical software is being developed to fill this methodological gap. It provides an extensible platform for pattern discovery that combines (1) *fuzzy logic–based conditions* applicable to numeric and linguistic variables, (2) *statistical evaluation of patterns* using hypothesis tests, (3) *interactive exploration and visualization* for human-guided discovery, and (4) *extensibility* for developing new pattern types and quality measures. Unlike previous tools, `nuggets` integrates the interpretability of linguistic rule systems with the flexibility of modern interactive Shiny-based data science interfaces. It thereby establishes a bridge between mathematical logic, fuzzy set theory, and practical data analysis within the R ecosystem.

Fuzzy Conditions and Linguistic Descriptions. The central concept in `nuggets` is the *condition*—a conjunction of predicates over the data. Conditions may be Boolean or fuzzy. Fuzzy predicates are constructed from numeric variables by membership functions (triangular, trapezoidal, raised-cosine) and evaluated via t-norms (Gödel, Goguen, Łukasiewicz). Each predicate corresponds to a linguistic label and defines a gradual transition between truth and falsity. This construction provides a mathematically rigorous link between data values and linguistic interpretation.

A particularly natural interpretation of fuzzy conditions appears in the tradition of *linguistic descriptions of data*. Here, discovered patterns are expressed as linguistic statements, e.g.,

"If income is high and age is middle, then occupation is likely IT."

Such expressions correspond formally to association rules or subgroup descriptions but use linguistic labels (*low, medium, high*) that mirror how humans reason about data. Because `nuggets` predicates are inherently fuzzy, they directly realize such linguistic terms, creating a

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bridge between the association-rule community and researchers developing linguistic summaries of data. In practice, this makes it possible to produce outputs that are both computationally precise and linguistically interpretable.

Mathematical and Statistical Foundations. The `nuggets` package follows the general logical–combinatorial approach of GUHA-style methods, where patterns are defined as logical conditions satisfied by subsets of data. Each condition is evaluated primarily through its *support*—the proportion or total membership degree of observations fulfilling it. Support thus serves as the key criterion for candidate generation and pruning, ensuring that discovered conditions describe sufficiently large and relevant parts of the dataset.

Beyond support, `nuggets` enables the computation of optional *quality measures* or statistical tests that quantify the strength or distinctiveness of discovered relations. These may include classical metrics such as confidence, lift, or correlation, as well as conditional comparisons of subpopulations. Currently, the package implements statistical evaluation for *two-sample contrasts* (e.g., differences in means or proportions) and *conditional correlation-based patterns* between numerical attributes. Fuzzy-weighted versions of such tests are planned for future releases but are not yet implemented in the current version.

Pattern Types Beyond Rules. At present, `nuggets` supports several families of pattern discovery tasks unified by the concept of conditions and support-based evaluation. The fundamental pattern type corresponds to *association rules* [7], representing implications of the form $A \Rightarrow C$ derived from frequent conditions. In addition to these, the package includes functionality for *conditional contrasts*, where subgroups are compared with respect to a target variable using statistical descriptors such as mean differences or correlation coefficients. These contrasts can be interpreted as a lightweight form of subgroup discovery or contrast pattern mining [2, 3, 5].

The design of `nuggets` is modular and extensible, allowing researchers to define new pattern types by specifying how candidate conditions are processed within a user-defined callback R function. Planned extensions include emerging patterns, exceptional models, and fuzzy-weighted statistical evaluation, which will further expand the analytical capabilities of the framework while maintaining interpretability and logical consistency.

Interactive Exploration and Visualization. Pattern discovery is inherently exploratory. The `explore()` interface, implemented via `Shiny`, enables the analyst to browse the discovered rules and visualize their statistical characteristics. Filtering allows users to focus on specific conditions or variables. This interactive environment turns pattern discovery into a human–machine dialogue rather than a static list of results.

Extensibility and Research Applications. A central goal of `nuggets` is to provide an open experimental environment for methodological innovation. The framework separates the mining process from the evaluation phase, allowing users to attach their own analytical logic to the discovered conditions. Specifically, every generated frequent condition can be passed to a *user-defined R function*, which may compute arbitrary statistics, quality measures, or model diagnostics. This mechanism enables the creation of entirely new pattern types—ranging from domain-specific contrasts to model-based exceptions—without modifying the internal search engine. This makes `nuggets` a genuine research framework rather than a fixed toolbox. It allows computer scientists, statisticians, and fuzzy logicians to experiment with new ideas in data-driven reasoning—testing how alternative fuzzification strategies, evaluation functions, or statistical measures influence the discovery of interpretable patterns. The framework thus serves as a bridge between theoretical exploration and practical implementation.

Example Workflow. A minimal example illustrates the integration of fuzzy preprocessing, statistical evaluation, and interactivity within the R session. As an example, a built-in **mtcars** dataset is used:

```
# Install the package (perform only once)
install.packages("nuggets")

# Load the library to the R environment
library(nuggets)

# Preprocess - dichotomize and fuzzify the numeric variables
cars <- mtcars |>
    partition(cyl, vs:gear, .method = "dummy") |>
    partition(carb, .method = "crisp", .breaks = c(0, 3, 10)) |>
    partition(mpg, disp:qsec, .method = "triangle", .breaks = 3)

# Search for associations
rules <- dig_associations(cars,
                          antecedent = everything(),
                          consequent = everything(),
                          max_length = 4,
                          min_support = 0.1,
                          measures = c("lift", "conviction"))

# Explore the found rules interactively
explore(rules, cars)
```

This workflow produces association rules based on both crisp and fuzzy predicates that are created from the original numeric variables of the **mtcars** dataset. The results can be visualized and inspected interactively, facilitating interpretation of association rules.

Conclusions and Outlook. The `nuggets` package provides a flexible and extensible framework for discovering interpretable data patterns based on frequent logical conditions. Its design unifies classical association-rule mining with linguistic and fuzzy representations, while enabling optional statistical evaluation for selected pattern types such as conditional contrasts and correlations. Pattern generation is driven by support, ensuring efficient mining of relevant conditions, whereas additional quantitative analyses or tests can be seamlessly attached when desired.

A major strength of `nuggets` lies in its extensibility. The framework allows users to define custom fuzzification schemes and to evaluate an arbitrary R function on every frequent condition, thereby enabling the creation of new, user-defined pattern types. This design encourages experimentation with alternative logical semantics, statistical measures, and application-specific evaluation criteria, making `nuggets` not only a tool for applied pattern discovery but also a research platform for developing new methods.

Future development will focus on incorporating fuzzy-weighted versions of statistical tests, expanding the library of built-in pattern types, and enhancing interactive visualization of complex fuzzy partitions. By combining formal clarity, methodological openness, and practical usability, `nuggets` contributes to the convergence of fuzzy logic, statistical reasoning, and interactive data exploration within the R environment. A detailed documentation of `nuggets` can be found at `https://beerda.github.io/nuggets/`.

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Fuzzy–Probabilistic Inference Systems Based on Piecewise Linear Weighted Quantiles

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1 Introduction and motivation

Fuzzy–probabilistic IF–THEN rule based systems form a class of knowledge-based systems where the antecedents encode fuzzy information and the consequents model probabilistic uncertainty present in the data. By integrating fuzzy set theory with probability theory, these systems offer a unified framework for managing both types of uncertainty and have been extensively investigated in the context of rule construction and inference design (see, e.g., [1, 2]).

In this work, we use a specific form of the IF-THEN rules and the inference mechanism, which coincide with the so-called *quantile fuzzy transform* (or L_1-fuzzy transform) that has been introduced and extensively investigated in [3, 4]. Particularly, given a suitable fuzzy partition $\Delta = \{A_1, \dots, A_m\}$ of a universe $\mathbb{U}$ and a random variable Y defined in a probability space $(\Omega, \mathcal{A}, P)$, the system takes the following formal form:

$$R_k : \text{IF } X \text{ is } A_k \text{ THEN } Y \sim Q_k(p), \quad k = 1 \dots, m, \tag{1}$$

where $Q_k(p)$ denotes the quantile function of the conditional distribution of Y given A_k. A crucial step in constructing the IF-THEN rules is the estimation $Q_k(p)$ from data. In [3], the quantile functions are derived from data using p-weighted quantiles, which are traditionally computed by the linear programming. In our recent work [5], we introduced an alternative and computationally efficient method for evaluating weighted quantiles derived from the analyzes of the right derivative of the corresponding convex objective function (see (2)). Nevertheless, prior research indicates that, despite their computational efficiency, classical weighted quantiles may be inadequate for accurately estimating the local positions of the output quantiles over the fuzzy inputs.

To address the limitation of the scalar weighted quantiles, we proposed to replace them by quantile piecewise linear functions [6], which better fit the output quantiles. This short paper presents a modification of the original method to enhance its applicability to forecasting tasks and includes also a comparison with the scalar weighted quantile approach.

2 Outline of the studied approach

Let us consider the fuzzy-probabilistic IF–THEN rules given in (1). The fuzzy sets A_k in the antecedents form a fuzzy partition Δ on $\mathbb{U}$ with nodes $c_1, \dots, c_m$ in which each A_k attains its

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maximum. The fuzzy partition covers $\mathbb{U}$, i.e., for any $x \in \mathbb{U}$, there exists $k = 1, \dots, m$ with $A_k(x) > 0$. Uniform or generalized fuzzy partitions are usually employed.

The empirical quantile functions $Q_k(p)$ in the consequents are estimated from data and represent the conditional distribution of Y given A_k. Particularly, let $\{(x_i, y_i)\}_{i=1}^n \subseteq \mathbb{U} \times \mathbb{R}$ be data, and let Δ be a fuzzy partition. For each A_k and $p \in [0,1]$, the *p-weighted quantile* $Q_k(p)$ is a real number z such that minimizes the following functional

$$\Phi_p^k(z) = \sum_{i=1}^n \rho_p(y_i - z) A_k(x_i), \quad \rho_p(u) = \begin{cases} p|u|, & u > 0, \\ (1-p)|u|, & u \le 0. \end{cases} \tag{2}$$

Observe that $A_k(x_i)$ serves as a weight for y_i; the closer x_k is to c_k, the higher the resulting degree. The vector $[Q_1(p), \dots, Q_m(p)]$ is known as the *direct Quantile Fuzzy transform* [3, 4]. Note that the weighted quantiles $Q_k(p)$ can be computed via linear programming [3] or the direct derivative-based method [5, 6]. The inference mechanism, which estimates the quantile function Q_x for any $x \in \mathbb{U}$, is defined as a weighted combination of quantiles Q_k (known as the *inverse QF-transform*):

$$Q_x(p) = \frac{\sum_{k=1}^m A_k(x) Q_k(p)}{\sum_{k=1}^m A_k(x)}. \tag{3}$$

In the following part, we generalize the scalar p-weighted quantiles to the piecewise linear p-weighted quantile functions associated with a *centering initial point* $(c_k, z_p) \in \mathbb{U} \times \mathbb{R}$ and extend the system introduced in (1). In practice, the centering initial point (c_k, z_p) corresponds to the k-th node in $\mathbb{U}$ and the p-weighted quantile $z_p = Q_k(p)$ is determined by $A_k \in \Delta$.

Let $p \in [0,1]$, $A_k \in \Delta$, and fix $(c_k, z_p) \in \mathbb{U} \times \mathbb{R}$. For any real numbers a_1, a_2, define the piecewise linear function

$$Q_{k,(a_1,a_2)}(p, x) = \begin{cases} a_1(x - c_k) + z_p, & x \le c_k, \\ a_2(x - c_k) + z_p, & x \ge c_k. \end{cases} \tag{4}$$

Define the objective functional

$$\Phi_p^k(a_1, a_2) = \sum_{i=1}^n \rho_p(y_i - Q_{k,(a_1,a_2)}(p, x_i)) A_k(x_i). \tag{5}$$

Then the *piecewise linear p-weighted function quantile associated with* (c_k, z_p) is $Q_{k,(\hat{a}_1,\hat{a}_2)}(p, \cdot) : \mathbb{U} \to \mathbb{R}$, where $(\hat{a}_1, \hat{a}_2)$ is any minimizer of Φ_p^k over all $(a_1, a_2) \in \mathbb{R}^2$. Similarly to the scalar case, the piecewise linear p-weighted quantile is not uniquely defined. The following lemma guarantees that such a quantile always exists.

Lemma 1 *For each $A_k \in \Delta$ and each centering initial point $(c_k, z_p) \in \mathbb{U} \times \mathbb{R}$, there exist a pair $(\hat{a}_1, \hat{a}_2)$ minimizing Φ_p^k over all $(a_1, a_2) \in \mathbb{R}^2$.*

Observe that we put if $z_p = Q_k(p)$, then $Q_k(p) = Q_{k,(\hat{a}_1,\hat{a}_2)}(p, c_k)$. Hence,

$$Q_{k,(\hat{a}_1,\hat{a}_2)}(p,c_k) \le Q_{k,(\hat{a}_1,\hat{a}_2)}(p', c_k), \quad \text{whenever } p \le p'.$$

Obviously, the same property is not true for an arbitrary $x \in \mathbb{U}$, i.e., $Q_k(\cdot, x) = Q_{k,(\hat{a}_1,\hat{a}_2)}(\cdot, x)$ is not a quantile function from its definition in general. To ensure that $Q_k(\cdot, x)$ is a quantile function for any $x \in \mathbb{U}$, we can proceed the following correction procedure:

$$Q_k^*(p, x) = \sup\{Q_k(q, x) \mid q \le p\}, \quad p \in [0,1].$$

Note that similarly, we can use the infimum or a mixture of both procedures with an initial p_0. In addition, we consider only a finite number of p-levels, therefore, the computation of $Q_k^*(p,x)$ is fast in practice. Now, the IF-THEN rules specified in (1) can be extended as follows:

$$R_k : \text{IF } X \text{ is } A_k \text{ THEN } Y(x) \sim Q_k^*(p,x), \quad k = 1, \dots, m, \tag{6}$$

where $Q_k^*(p,x)$ expresses (locally) the conditional quantile function given A_k at the point $x \in \mathbb{U}$. The inference mechanism for this new type of rule based system is given for each $x \in \mathbb{U}$ as follows

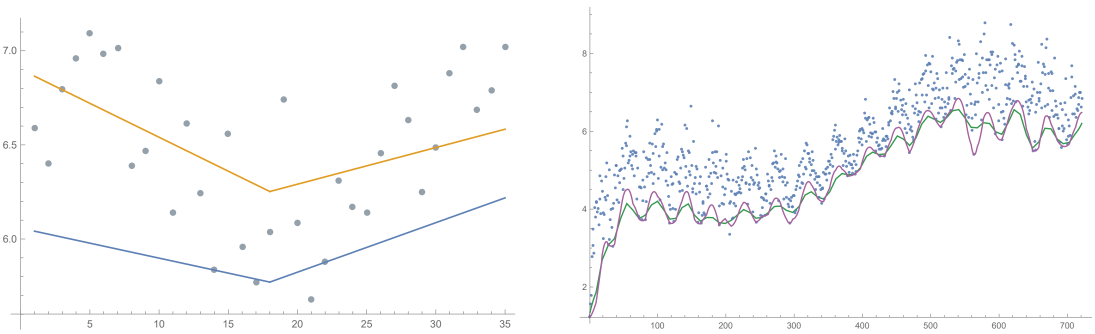
$$Q_x(p) = \frac{\sum_{k=1}^m A_k(x) Q_k^*(p,x)}{\sum_{k=1}^m A_k(x)}. \tag{7}$$

3 Application in time series analysis and forecasting

In this section, we illustrate the application of piecewise linear weighted quantiles within a fuzzy–probabilistic inference system (FPIS) using a synthetic time series of $n = 721$ observations (x_i, y_i), generated by

$$y_i = g(x_i) + \varepsilon(x_i), \quad g(x) = \tfrac{1}{2}\sin\left(\tfrac{x}{7}\right) + \cos\left(\tfrac{x}{90}\right) + \ln(x+1),$$

where $\varepsilon(x) \sim \mathcal{N}(0, \sigma(x))$, $\sigma(x) = 0.4 + \frac{1}{6}\sin\left(\frac{\pi x}{240}\right)$. We employ a uniform fuzzy partition $\Delta = \{A_1, \dots, A_{81}\}$ with an offset of 2 (the support of each fuzzy set is twice the shift length). In Fig. 1, panel (a) displays the conditional quantile functions $Q_k^*(p,x)$ given A_{53} for fixed $p \in \{0.05, 0.75\}$, while panel (b) compares the resulting quantile functions of the rule systems (1) (shown in green) and (6) (shown in violet) along $\mathbb{U}$ for the fixed value $p = 0.05$. From a



(a) (b)

Figure 1: Illustration of piecewise linear and scalar weighted quantile based FPIS

visual comparison, one can observe that the application of piecewise linear weighted quantile functions yields more accurate estimations of the p-quantile functions along the universe $\mathbb{U}$. This observation is further supported by the Bhattacharyya distance between the theoretical and estimated (normally distributed) density functions, see Fig. 2.

Without going into details, the extended FPIS can be used to forecast probability distributions. To illustrate this, we employ a simple autoregressive model. First, we consider $\mathbb{U} = [0, T]$ and extend the fuzzy partition Δ to Δ^+ by adding fuzzy sets $A_{m+1}, \dots, A_{m+v}$. Then, using the AR(ℓ) model, we estimate the parameters of the piecewise linear weighted quantile functions $(a_{1,m+i}, a_{2,m+i}, y_{m+i})$ for $i = 1, \dots, v$, and apply the inference mechanism given in (7) to obtain the quantile functions for future time points.

In Fig. 3(a), the forecasted p-quantiles for 45 future time points are shown, while Fig. 3(b) displays the ARMA models computed and trained on scalar quantiles in *Mathematica*. The forecast based on the FPIS outperforms almost a linear forecast provided by ARMA modeling.

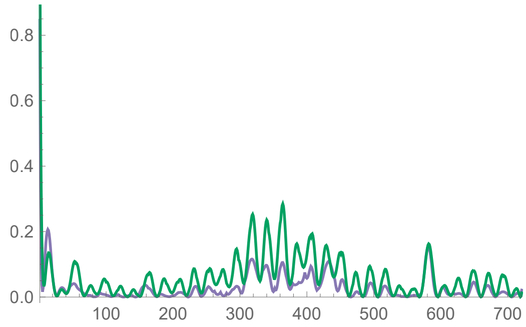
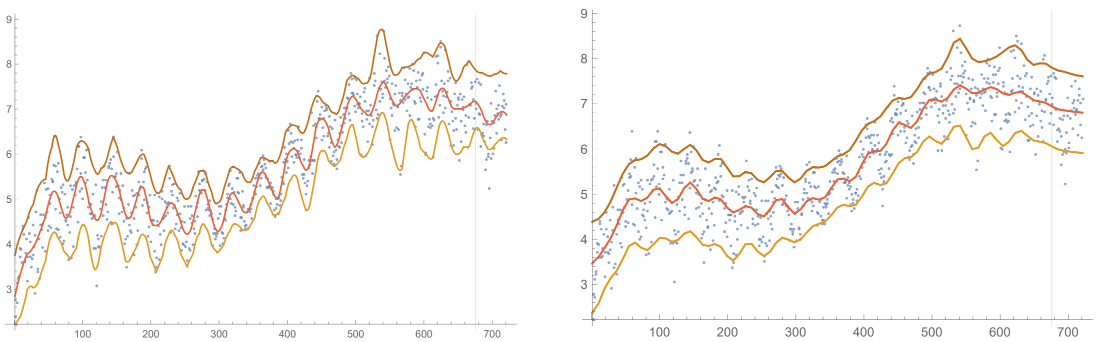


Figure 2: Bhattacharyya distance between the theoretical normal density function and its estimates obtained using the piecewise linear (violet) and scalar (green) weighted quantiles.



(a) (b)

Figure 3: Forecasting for $p \in \{0.05, 0.5, 0.95\}$: (a) piecewise linear quantiles; (b) ARMA.

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On Data–Driven Fuzzy Partition in the Fuzzy–Probabilistic Inference System Framework

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1 Preliminaries and motivation

This paper is focused on *fuzzy–probabilistic IF–THEN rules based systems*, where the antecedents encode fuzzy information and consequents represent probability distributions of the output variable. This system, which combines both types of uncertainty in one framework, can be applied effectively in time series analysis and forecasting. Let $\mathbb{U}$ be a universe, let $\Delta = \{A_1, \ldots, A_m\}$ be a fuzzy covering of $\mathbb{U}$, i.e., for every $x \in \mathbb{U}$, at least one $A_k(x) > 0$, and Y denote a random variable defined in a probability space $(\Omega, \mathcal{A}, P)$. Then, the rules take the form (see, [1]):

$$R_k : \text{ IF } X \text{ is } A_k \text{ THEN } Y \sim Q_k(p), \quad k = 1, \ldots, m, \tag{1}$$

where $Q_k(p)$ denotes the quantile function of the conditional distribution of Y given A_k. It is worth noting that, in practice, uniform or generalized fuzzy partitions are typically constructed by shifting equidistant fuzzy sets along the domain axis. The quantile functions in the antecendents of rules in (1) are estimated from data as follows. Assume a dataset $\{(x_i, y_i)\}_{i=1}^n \subset \mathbb{U} \times \mathbb{R}$ and let Δ denote a fuzzy covering of $\mathbb{U}$. For any $p \in [0, 1]$, the quantile function $Q_k(p)$ is defined as the p–weighted quantile of the pairs $(y_1, A_k(x_1)), \ldots, (y_n, A_k(x_n))$. In particular, the p–weighted quantile is the value $z_p \in \mathbb{R}$ that minimizes the functional

$$\Phi_p(z) = \sum_{i=1}^{n} \rho_p(y_i - z)\, A_k(x_i), \quad \rho_p(u) = \begin{cases} p|u|, & u > 0, \\ (1-p)|u|, & u \le 0. \end{cases} \tag{2}$$

The inference mechanism, which estimates the quantile function $Q_x(p)$ for any $x \in \mathbb{U}$, is defined as the weighted average of quantiles $Q_k(p)$:

$$Q_x(p) = \frac{\sum_{k=1}^{m} A_k(x) Q_k(p)}{\sum_{k=1}^{m} A_k(x)}. \tag{3}$$

Note that this is not the only way to define the inference mechanism for (1); several alternatives can be found in [2].

The effectiveness of fuzzy–probabilistic inference systems (FPIS) has been demonstrated in various applications, for example, in time series analysis [2]. However, several aspects of such systems remain open and challenging, particularly the construction of an appropriate fuzzy

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partition. In practice, uniform fuzzy partitions are commonly used because of their simplicity, yet they fail to capture the complex structures hidden in the data. This raises a key question: can we construct a data-driven fuzzy partition that better reflects local behavior under a well-defined criterion? This short paper introduces three methods based on different criteria for designing non-uniform, data-dependent fuzzy partitions, presented in algorithmic form. A detailed theoretical analysis of these methods is left for future work.

2 Data-driven fuzzy partitions

Criterion 1: Limited range size for data values. Assume, for simplicity, that the dataset is given by $\{(x_i, y_i)\}_{i=1}^n \subset \mathbb{U} \times \mathbb{R}$, where $x_i = i$. Let $\varepsilon > 0$ be a predefined threshold. For a finite vector $\bar{y} = (y_1, \ldots, y_k)$, recall that its *range* is defined as

$$R(\bar{y}) = \max\{y_1, \ldots, y_k\} - \min\{y_1, \ldots, y_k\}.$$

We construct consecutive subvectors $\bar{y}_1, \ldots, \bar{y}_n$, where each $\bar{y}_k = (y_k, \ldots, y_{k+m_k})$ is determined such that

$$R(\bar{y}_k) \leq \varepsilon \quad \text{and} \quad R(\bar{y}_k^+) > \varepsilon, \tag{4}$$

where $\bar{y}_k^+ = (y_k, \ldots, y_{k+m_k+1})$. Each vector $\bar{y}_k$ is associated with an interval $I_k = [k, k + m_k]$, which serves as the *support* of the fuzzy set A_k in the resulting fuzzy partition. The node of A_k is placed at the midpoint of I_k. Since not all constructed intervals are useful for the final partition, we propose three *selection strategies* to determine the next interval $I_{k_{\ell+1}}$, given the previously selected ℓ intervals $I_1, \ldots, I_{k_\ell}$. The next index $k_{\ell+1}$ is chosen based on $\bar{y}_{k_\ell}$ as follows:

(i) **Extrema-based:** choose the smaller of the indices corresponding to the minimum and maximum elements of $\bar{y}_{k_\ell}$;

(ii) **Median-based:** choose the index corresponding to the (classical) median of $\bar{y}_{k_\ell}$, ensuring that consecutive supports do not overlap in median values;

(iii) **Weighted median-based:** choose the index corresponding to the *weighted median* of $\bar{y}_{k_\ell}$, where the weights are induced by the membership degrees $A_{k_\ell}(i)$ for $i = k_\ell, \ldots, k_\ell + m_{k_\ell}$.

Note that the threshold ε must be appropriately adjusted to obtain a reasonable number of fuzzy sets in the resulting fuzzy partition. Obviously, a smaller value of ε results in a higher number of fuzzy sets, and vice versa.

Example 1 *Consider synthetic data* $y_i = g(i)$ *with*

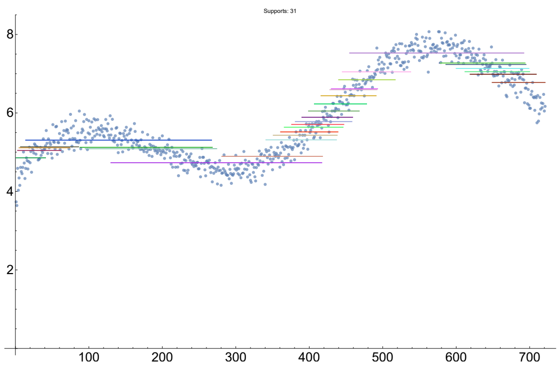
$$g(x) = \ln(x + 20) + \cos(x/90) + \tfrac{1}{3}\sin\left(\frac{x\log(x + 50)}{200}\right) + \upsilon(x),\ \upsilon(x) \sim \mathcal{N}(0, 0.2^2).$$

Let $\varepsilon = 1.7$. Figure 1 illustrates the supports obtained from the three starting-index methods and the corresponding moving median computed via L_1-based minimization [2]. The moving median exhibits favorable behavior, aligning well with the observed data pattern.

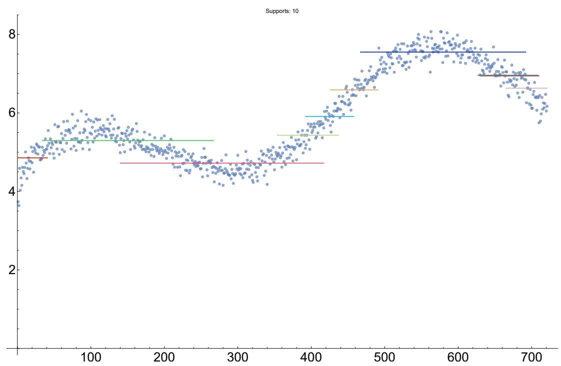
Criterion 2: Limited variance of data values. This criterion is a modification of the previous one, in which the range is replaced by the variance. Specifically, condition (4) is replaced by

$$\mathrm{Var}(\bar{y}_k) \leq \varepsilon \quad \text{and} \quad \mathrm{Var}(\bar{y}_k^+) > \varepsilon, \tag{5}$$

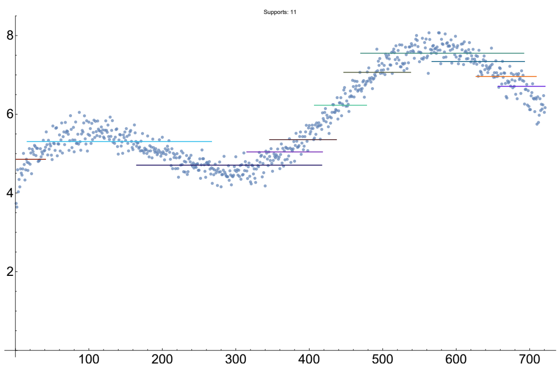
where the variance of the vector components is computed in the standard way. All remaining steps, including the selection strategies, are identical to those described earlier. Figure 2 illustrates the result for Example 1 with $\varepsilon = 0.3$ and the weighted-median-based strategy.



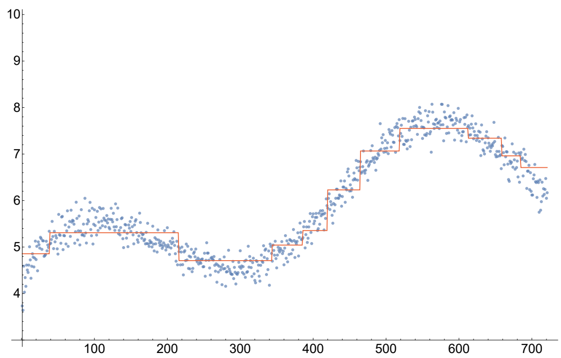
(a) Extrema-based strategy



(b) Median-based strategy



(c) Weighted median-based strategy



(d) Moving median based on the third strategy

Figure 1: Resulting fuzzy set supports based on Criterion 1 and the moving median.

Criterion 3: Relative error of variances of local linear regression residuals. The third strategy evaluates variance stability by analyzing the residuals of linear regressions across intervals and their subintervals. The domain is partitioned by recursive bisection whenever significant relative changes in variance are detected.

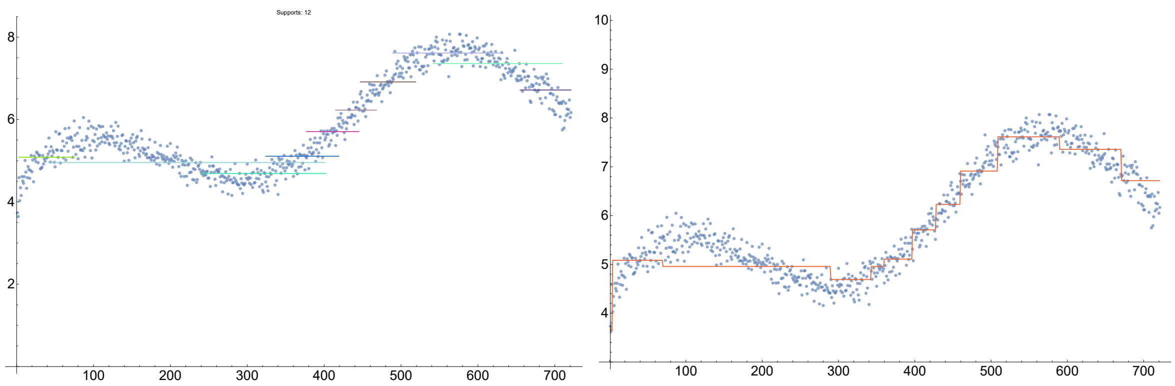
Let I denote an interval (e.g., $I = [1, n]$) and let σ_I denote the standard deviation of the residuals for the linear regression model over the interval I. We define the *relative error of residual variances* over the interval I (with $\sigma_I > 0$) and its subinterval J as

$$R_{I,J} = \frac{\sigma_I - \sigma_J}{\sigma_I}. \tag{6}$$

Denote by $\bar{x}_I$ the mean, by $V_I = \sigma_I / \bar{x}_I$ the coefficient of variation, and by n_I the number of integers in I. Let $R_{\min}$, $V_{\min}$, and $n_{\min}$ be the minimum thresholds for the relative error, the coefficient of variation, and the interval size, respectively.

The partition procedure recursively divides each interval into two half-subintervals according to the following rule. Assume that an interval I has been divided by the partition procedure into two half-subintervals. Let $I/2$ denote the left (or right) subinterval whose right (or left) endpoint coincides with the midpoint of I, where the endpoints are appropriately rounded to consecutive integers. The subinterval $I/2$ is further divided into its two half-subintervals if $n_{I/2} > n_{\min}$ and either of the following conditions holds:

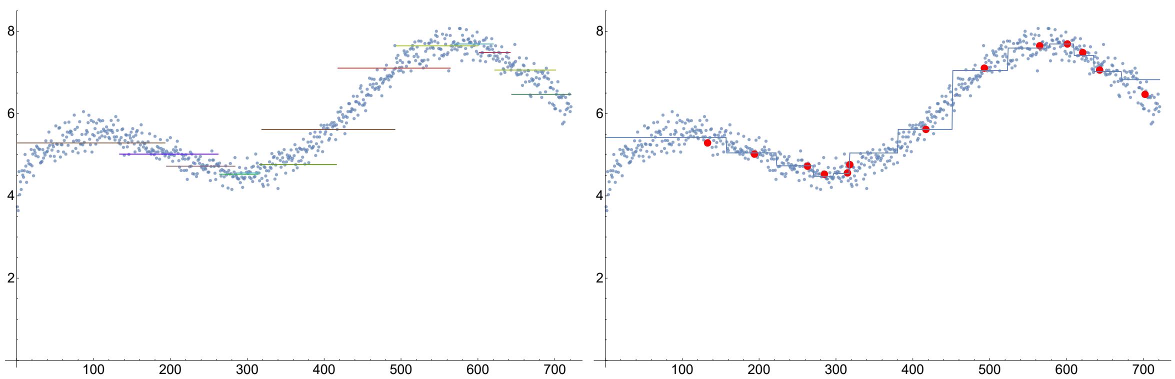
$$R_{I,I/2} > R_{\min} \quad \text{or} \quad V_{I/2} > V_{\min}. \tag{7}$$



(a) Weighted median-based strategy (b) Moving median based on the third strategy

Figure 2: Resulting fuzzy set supports based on Criterion 2 and the moving median.

When $I/2$ is further subdivided and the bisection procedure is invoked with new parameters, the linear regression is recomputed on the same dataset. Hence, σ_J can be passed as a parameter to replace σ_I in the next step. After finitely many steps, we obtain a disjoint family of subintervals $\{I_j\}$ of $I = [1, n]$. To construct overlapping supports for the fuzzy partition, each I_j is extended by redefining its endpoints as the indices of the medians of values in the adjacent intervals I_{j-1} and I_{j+1} (cf. Median-based selection strategy). Figure 3 illustrates this procedure on data from Example 1, with red bold dots indicating medians within intervals.



(a) Resulting supports (b) Moving median based on the received supports

Figure 3: Resulting fuzzy set supports based on Criterion 3 and the moving median.

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On Inference Mechanisms of Fuzzy-Probabilistic Inference Systems

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1 Outline of the studied problem

This work concerns the inference mechanism of *fuzzy–probabilistic inference systems* (FPIS), a special class of rule-based models in which antecedents encode fuzzy information while consequents represent conditional probability distributions of the output random variable. Such systems are particularly well suited for modeling uncertainty in time series responses. Specifically, we study a system of m rules taking the form $(k = 1, \dots, m)$ [2]:

$$R_k := \text{IF } X \text{ is } A_k \text{ THEN } Y \text{ follows the probability distribution given by } Q_k(q), \tag{1}$$

where each consequent $Q_k(q)$, for $q \in [0, 1]$, is an *empirical quantile function* representing the underlying probability measure (distribution) of the output random variable Y conditioned on the fuzzy set A_k defined on a given universe X. We assume familiarity with basic probabilistic notions such as probability measures, cumulative distribution functions, and quantile functions.

In our work, fuzzy sets A_k used in the antecedents form a family Δ that covers the universe X, i.e., for every $x \in X$ there exists an $A_k \in \Delta$ with $A_k(x) > 0$. Recall that the fuzzy sets $A_1, \dots, A_m \in \Delta$ are defined on the nodes $c_1, \dots, c_m$, respectively, where each A_k attains its maximum. In practice, we often consider families of fuzzy sets that form a simple or generalized uniform fuzzy partition, see Fig. 1.

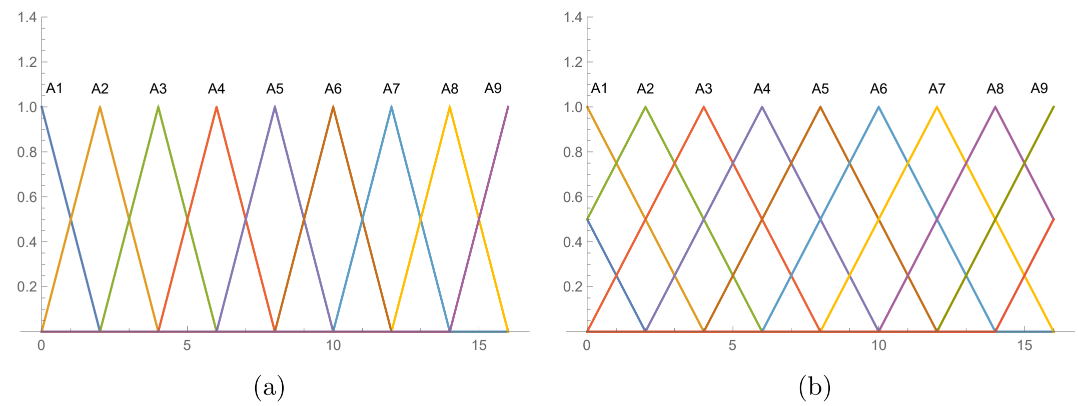

Figure 1: Examples of uniform fuzzy partitions: (a) simple, (b) generalized.

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Given a dataset $\{(x_i, y_i)\}_{i=1}^n \subseteq X \times \mathbb{R}$ and a generalized partition Δ, the q-weighted quantile $Q_k(q)$ associated with $A_k \in \Delta$ is obtained as the minimizer of (cf. [1]):

$$\Phi_q(z) = \sum_{i=1}^{n} \rho_q(y_i - z)\, A_k(x_i), \quad \rho_q(u) = \begin{cases} q|u|, & u > 0, \\ (1-q)|u|, & u \leq 0. \end{cases} \tag{2}$$

The inference mechanism that produces an empirical quantile function for an arbitrary input $x \in X$ is defined as a linear combination of the local quantile functions $Q_k(q)$.

Definition 1 *[1, 2] Let $\{(x_i, y_i)\}_{i=1}^n \subseteq X \times \mathbb{R}$ be given, together with a fuzzy partition $\Delta = \{A_1, \ldots, A_m\}$ of X. For each k, let Q_k be the weighted quantile function associated with A_k via the k-th rule in (1). Then, for any $x \in X$, the quantile function $Q_x : [0,1] \to \mathbb{R}$ is defined as*

$$Q_x(q) = \sum_{k=1}^{m} w_k(x) Q_k(q), \tag{3}$$

where $w_k(x) = A_k(x) / \sum_{j=1}^m A_j(x)$, $k = 1, \ldots, m$, denotes the normalized weight.

2 Inference mechanisms: theoretical foundation

Fuzzy rule-based systems express the relationship between two variables in a rough manner, while the inference mechanism provides a means to refine this rough relationship into a complete relationship, which can be used in practice. In [7], we experimentally analyzed and compared the standard weighted average of quantile functions as the inference mechanism for FPIS and other alternatives on synthetic and real datasets. This work left open a deeper theoretical analysis, particularly regarding the motivation and justification of the inference mechanisms used, including the original weighted average - all of which were introduced in a largely ad hoc manner. This knowledge gap motivates us to a deeper investigation of the foundation of the inference mechanism for FPIS, in which one part of the IF–THEN rules follows a fuzzy framework, while the other adopts a probabilistic one. For this purpose, we use the Wasserstein metric space $(\mathcal{P}_p(\mathbb{R}), W_p)$ of the order $p > 0$ on $\mathbb{R}$ (see, [4]), where

$$\mathcal{P}_p(\mathbb{R}) = \left\{ \mu \text{ is a probability measure on } (\mathbb{R}, \mathcal{B}(\mathbb{R})) : \int_{\mathbb{R}} |x|^p \, d\mu(x) < \infty \right\}$$

and W_p is the p-Wasserstein distance given for any probability distribution μ and ν as follows

$$W_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x - y|^p \, d\pi(x,y) \right)^{1/p},$$

where $\Pi(\mu, \nu)$ denotes the set of all couplings of μ and ν, i.e. the set of all probability measures on $\mathbb{R} \times \mathbb{R}$ with marginals μ and ν. Note that a probability measure μ on $\mathbb{R}$ is defined on the Borel algebra on $\mathbb{R}$ and can be equivalently expressed by the respective probability mass function f_μ, cumulative distribution function F_μ or quantile function Q_μ. For example, we have $F(x) = \mu((-\infty, x))$ and $Q_\mu(q) = F^{-1}(q) = \inf\{x \in \mathbb{R} : F(x) \geq q\}$. The p-Wasserstein distance for probability measures on $\mathbb{R}$ can be simply expressed in terms of quantile functions as follows

$$W_p(\mu, \nu) = \left(\int_0^1 |Q_\mu(q) - Q_\nu(q)|^p \, dq \right)^{1/p} = \|Q_\mu - Q_\nu\|_{L_p(0,1)},$$

In what follows, we show that the inference mechanism for FPIS, originally defined ad hoc as the weighted average of quantile functions, can be rigorously formulated using the 2-Wasserstein distance. Specifically, it corresponds to the Wasserstein barycenter, also known as the Fréchet mean. This observation allows us to introduce a broader class of inference mechanisms based on p-Wasserstein distances. For convenience, we assume a generalized fuzzy partition $\Delta = \{A_1, \ldots, A_m\}$ of X, and define $w_k(x) = A_k(x)/\sum_{j=1}^m A_j(x)$ for any $k = 1, \ldots, m$ and $x \in X$.

Inference mechanism as the Wasserstein barycenter in $\mathcal{P}_2(\mathbb{R})$

Let $\mu_1, \ldots, \mu_m \in \mathcal{P}_2(\mathbb{R})$ with weights $w_i \geq 0$ such that $\sum_{i=1}^m w_i = 1$. The *Wasserstein barycenter* (or *Fréchet mean*) of $\mu_1, \ldots, \mu_m$ is a probability measure $\gamma \in \mathcal{P}_2(\mathbb{R})$ given as

$$\gamma = \arg \min_{\eta \in \mathcal{W}_2(\mathbb{R})} \sum_{i=1}^m w_i W_2^2(\eta, \mu_i).$$

It is well known that the Wasserstein barycenter of probability measures on $\mathbb{R}$ always exists and admits a closed-form expression in terms of quantile functions:

$$Q_\gamma(q) = \sum_{i=1}^m w_i Q_{\mu_i}(q), \tag{4}$$

where Q_{μ_i} denotes the quantile function of μ_i. Thus, the inference rule (3) coincides with the Wasserstein barycenter of empirical probability measures, which motivates the following equivalent formulation of the inference mechanism.

Definition 2 (Inference mechanism as the Fréchet mean) *Let $\{(x_i, y_i)\}_{i=1}^n \subseteq X \times \mathbb{R}$ be given data, and let Δ be a generalized fuzzy partition. We define $f^\star : X \to \mathcal{W}_2(\mathbb{R})$ as*

$$f^\star(x) = \arg \min_{\gamma \in \mathcal{P}_2(\mathbb{R})} \sum_{k=1}^m w_k(x)\, W_2^2(\gamma, \mu_k), \tag{5}$$

where $(\mu_k)_{k=1}^m$ are the probability measures with the empirical quantile functions $(Q_k)_{k=1}^m$ associated with the fuzzy sets $(A_k)_{k=1}^m$.

By the uniqueness of the Wasserstein barycenter in $\mathcal{P}_2(\mathbb{R})$, $f^\star(x)$ is well-defined, and its quantile function is given by (4). Assume that X is a metric space denoted as (X, d). The following proposition shows a local convergence property of f^* at nodes c_k of fuzzy sets $A_k \in \Delta$.

Proposition 3 *Let (X, d) be a metric space, and let $\Delta = \{A_1, \ldots, A_m\}$ be a generalized fuzzy partition such that each $A_k \in \Delta$, $k = 1, \ldots, m$, is K-Lipschitz continuous. Assume that the quantile function $Q_k \in L_2(0,1)$ associated with $A_k \in \Delta$ satisfies $\|Q_k\|_{L_2(0,1)} \leq B$ for any $k = 1, \ldots, m$. Moreover, assume that $S(x) = \sum_{i=1}^m A_k(x) \geq S_0 > 0$ for every $x \in X$. Then for any c_k and $x \in X$, we have*

$$W_2\big(f^\star(x), f^\star(c_k)\big) \leq \frac{2mBK}{S_0}\, d(x, c_k).$$

Inference mechanism as the Wasserstein barycenter in $\mathcal{P}_p(\mathbb{R})$

The observation in the previous paragraph suggests a natural generalization of the inference mechanism in which the case $p = 2$ is extended to an arbitrary p. Particularly, the function $f_p^\star : X \to \mathcal{W}_p(\mathbb{R})$ is defined as the Wasserstein barycenter of $\mu_1, \dots, \mu_m$ in $\mathcal{P}_p(\mathbb{R})$ for which

$$f_p^\star(x) \in \arg \min_{\gamma \in \mathcal{P}_p(\mathbb{R})} \sum_{k=1}^{m} w_k(x)\, W_p^p(\gamma, \mu_k). \tag{6}$$

Unlike the case $p = 2$, the Wasserstein barycenter is generally not unique for arbitrary p. To define $f_p^*(x)$ properly, one must select a specific solution. For example, for $p = 1$ (the Wasserstein barycenter coincides with the Fréchet median [5]), a minimizing probability measure $\gamma_x = f_1^\star(x)$ has a quantile function

$$Q_{\gamma_x}^*(q) \in \arg\min_{z\in\mathbb{R}} \sum_{k=1}^{m} w_k(x)|z - Q_k(q)|,$$

where the set of minimizers for each q may form a closed interval. To guarantee the uniqueness of f_1^*, we select the left boundary (a canonical choice) of the closed interval of minimizers.

Definition 4 (Inference mechanism as the Fréchet median) *Let $\{(x_i, y_i)\}_{i=1}^n \subseteq X \times \mathbb{R}$ be given data, and let Δ be a generalized fuzzy partition. We define the function $Q^* : X \times [0, 1] \to \mathbb{R}$ as follows*

$$Q^*(x, q) = \min\Big\{\arg\min_{z\in\mathbb{R}} \sum_{k=1}^{m} w_k(x)\ |z - Q_k(q)|\Big\},$$

where $(Q_k)_{k=1}^m$ denote the empirical quantile functions associated with the fuzzy sets $(A_k)_{k=1}^m$. The function $f_1^* : X \to \mathcal{W}_1(\mathbb{R})$ is defined by $f_1^*(x) = \gamma_x$ which quantile function is $Q^*(x, -)$.

The general definition of the inference mechanism for FPIS via the Wasserstein barycenter in $\mathcal{P}_p(\mathbb{R})$ provides a theoretical framework for further investigation of its properties, such as the local approximation property formulated in Lemma 3. Moreover, applying these theoretical results to practical implementations of the inference mechanism is an important direction for future research.

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A General Framework for Multiplets Selection: Algorithmization and Complexity Analysis

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Abstract

In this contribution, we present the `multiplets` algorithm for constructing and selecting optimal sets of disjoint hyperedges across multiple groups in tabular data. We describe its main computational steps and provide a complexity analysis covering both the edge construction and optimization phases, based on the Linear Sum Assignment method and the Constraint Programming SAT-based solver.

1 Introduction

Forming balanced and representative teams across several predefined groups is a common problem in many areas of data analysis, education, and optimization. Typical applications include matching individuals from different categories or cohorts so that each resulting team (also called *multiplet*) contains one member from each group and the overall configuration is as coherent as possible according to some similarity or cost measure. This task can be viewed as finding a set of disjoint *hyperedges* in a multipartite graph, where vertices represent individuals and edge weights represent pairwise similarities or distances [1, 2]. Such matching problems appear not only in team assignment, but also in cross-domain record linkage, cohort pairing, or multi-modal data alignment [3, 4].

To address this, we introduce the `multiplets` algorithm, which systematically constructs and selects an optimal set of disjoint multiplets across multiple groups in tabular data. The method generalizes bipartite matching to a multipartite setting and combines efficient data manipulation (through the Polars framework) with exact optimization using either the Linear Sum Assignment method or the Constraint Programming SAT-based solver from OR-Tools. The following example illustrates the core idea in an intuitive context. For practical implementation we developed the `multiplets` Python module [5], which supports partitioning a dataframe into groups, computing pairwise edges and hyperedges, and selecting optimal set of disjoint hyperedges via exact optimization.

Imagine you have several groups of students (for example, different classes or training groups). The goal is to form *multiplets* — small teams where each member comes from a different group, and the members of the team are chosen so that they "fit together" according to some criterion (for example, similar performance, body height, or test score).

Our algorithmic approach to solve this problem consists of three main steps:

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1. **Score pairs across groups:** For every possible pair of students from different groups, the algorithm computes how similar they are (a "weight"). Pairs that do not meet the condition (for example, weight is too large) are discarded.

2. **Build multiplets:** From these pairs the algorithm constructs all possible teams (multiplets), always including one student from each group.

3. **Optimize the selection:** The algorithm then uses a suitable optimization method to find the best set of disjoint multiplets (i.e. the largest set with the smallest total weight).

So in simple words: the result should help to divide students into balanced, mixed teams.

In more detail, the procedure first verifies the validity of the input dataframe, then constructs pairwise edges and corresponding hyperedges across all groups, and finally applies an exact optimization step—using either the Linear Sum Assignment method or the Constraint Programming SAT-based solver—to select the optimal set of disjoint multiplets.

2 Description of the Multiplets Algorithm

The `multiplets` module implements functions for constructing, weighting, and selecting sets of *hyperedges* (multiplets) in a multipartite graph derived from tabular data.

- **Initialization:** Given a Polars `DataFrame` containing a unique identifier column (`id`), a grouping attribute (`group`) and other columns, the constructor partitions the vertices according to their group membership using the `partition_by` function. It validates the input and raises an error if any of the following conditions hold: the dataframe is empty, required columns are missing, identifiers are non-unique or null, or the number of groups is less than 2 or exceeds 10 (unless `check_too_big=False`). Each partition is internally indexed by an integer column for efficient joining.

- **Edges and hyperedges construction (`init_edges`):** For every pair of groups, all possible inter-group edges are generated via a Cartesian product. A user-specified `weight` expression (column name, numeric literal, or Polars expression from the other columns) is evaluated for each pair and stored in column `_weight`. An optional `filter` condition removes edges that do not satisfy a constraint (e.g., an upper bound for `_weight`). Hyperedges are then constructed as complete combinations of vertices containing exactly one vertex from each group, such that all required pairwise edges exist. The total weight of a hyperedge is obtained by an aggregation function (defaulty by `sum_horizontal`) of the weights of the pairwise edges. The resulting dataframe `hyperedges` is deterministically sorted by vertex identifiers.

- **Selection of optimal multiplets (`find_multiplets`):** The algorithm searches for a set of disjoint hyperedges that maximizes the number of multiplets (primary objective) and minimizes the total weight (secondary objective). Parameter `penalty` can be used to prefer smaller set of multiplets when the weights in a bigger set are too large. Two solvers are implemented:

  - For **two groups**, the Linear Sum Assignment (**LSA**) algorithm from OR-Tools is used by default. Algorithm LSA works only with integer weights, so the weights are multiplied by an optional parameter `multiplier` to reduce rounding errors. Algorithm LSA searches for a perfect matching in a graph, therefore some virtual edges are internally added.

  - For **three or more groups** (or when `force_CPSAT=True`), a Constraint Programming formulation is solved by the OR-Tools **CP-SAT** optimizer. Each candidate hyperedge corresponds to a binary variable, and mutual exclusivity between overlapping hyperedges is enforced as linear constraints. The objective function maximizes the sum of (`penalty`−`_weight`) terms. Optional parameters include `max_time` which forces the solver to stop after a given time, returning a sub-optimal result.

  The method returns the Polars `DataFrame` with selected multiplets. If no feasible solution is found, a `multipletsErrorNoSolution` is raised.

- **Reproducibility:** The current implementation of `find_multiplets` is not deterministic for both solvers and the results are not reproducible if there is more than one optimal solution, which is often the case. This is caused by stochastic effects in OR-Tools heuristics.

In summary, the `multiplets` module constructs a multipartite hypergraph from tabular data, evaluates feasible cross-group combinations, and employs exact optimization (LSA or CP-SAT) to obtain a maximum set of disjoint hyperedges with minimal total weight. This framework provides a reproducible and interpretable approach to structured matching across multiple categorical partitions.

The `multiplets` module also contains several functions for easier tabular manipulation with the dataframe with selected multiplets, and with other dataframes.

3 Computational Complexity

Let the input consist of k groups with sizes $n_0, \ldots, n_{k-1}$ and total size $N = \sum_{i=0}^{k-1} n_i$. Denote by $M = \sum_{0\le i<j\le k-1} m_{ij}$ the total number of *kept* pairwise edges after filtering, with $m_{ij} \le n_i n_j$, and by $H \le \prod_{i=0}^{k-1} n_i$ the number of feasible hyperedges (complete k-cliques across groups).

(1) Partitioning and input checks. Building group partitions and validating input (non-empty dataframe, required columns, uniqueness of `id`, and bounds on k) runs in

$$\text{time } O(N), \qquad \text{space } O(N).$$

(2) Pairwise edge construction (init_edges). For each pair (i, j) the algorithm performs a Cartesian product, evaluates the `weight` expression, and applies the `filter`. Let m_{ij} be the number of kept edges for (i, j). Then

$$\text{time } O\Big(\sum_{i<j} n_i n_j\Big) \text{ (worst case) or } O(M) \text{ (after filtering)}, \qquad \text{space } O(M).$$

(3) Hyperedge (clique) construction. The pairwise edge tables are joined to form cliques with exactly one vertex from each group; the $\binom{k}{2} = O(k^2)$ pair weights are aggregated horizontally (by default, by sum). With H feasible hyperedges, we have

$$\text{time } O(Hk^2) \text{ for aggregation} + \text{(hash-join overhead)}, \qquad \text{space } O(H).$$

In dense settings, H can approach $\prod_i n_i$; in practice, filtering controls H.

(4) Selection (find_multiplets).

- **LSA algorithm** ($k = 2$ unless `force_CPSAT=True`): after augmenting to a square weight matrix of size $p = n_0 + n_1$, the Hungarian (LSA) algorithm yields

$$\text{time } O(p^3), \qquad \text{space } O(p^2),$$

with $O(p^2)$ to build the augmented table and $O(p)$ to extract matches.

- **CP-SAT algorithm** ($k \geq 3$ or `force_CPSAT=True`): the CP-SAT model has one binary variable per hyperedge (H total) and $\leq N$ at-most-one constraints (one per person), where each constraint can touch up to $\prod_{j \neq i} n_j$ hyperedges in the dense worst case. Model building is

$$\text{time } O(Hk) + O(\text{nnz}), \qquad \text{space } O(H + \text{nnz}),$$

where nnz = number of nonzero coefficients in all constraints; and the solve time is exponential in the worst case (NP-hard set-packing); CP-SAT performs well empirically but has no polynomial worst-case bound. An optional time limit `max_time` caps the solve time.

(5) Summary. The build phase costs $O\big(\sum_{i<j} n_i n_j\big)$ time and $O(M)$ space for edges, then $O(Hk^2)$ time and $O(H)$ space for hyperedges (up to join constants). Selection is $O((n_0 + n_1)^3)$ for $k = 2$ (after $O(n_0 n_1)$ build) and NP-hard for $k \geq 3$ via CP-SAT. Aggressive filtering that reduces m_{ij} (hence M) typically shrinks H dramatically and dominates practical scalability.

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Properties of Graded Peterson's Square of Opposition as Immediate Inferences

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Abstract

Immediate inferences are arguments where the conclusion is supported by just one premise. There are several ways to infer a conclusion from a premise. We can use conversion, obversion, contraposition or the properties of some structure of opposition. In this article, we will focus on the study of immediate inference for several forms of intermediate quantifiers that form a graded Peterson's square of opposition which describes properties of *contrary, contradictory, subcontrary, subaltern*, and *superaltern* between these quantifiers.

Keywords: Generalized Intermediate quantifiers; Graded Peterson's square of opposition; Immediate inference

1 Introduction

We use quantifiers to quantify the number of objects in everyday communication. In classical logic, we distinguish universal quantifier *All* and particular quantifier *Some*. Classical square of opposition has been explored in several publications including [1, 2, 3]. Square of the opposition contains classical quantifiers which we denote by capital letters **A,E,I,O** and describes properties of *contrary, contradictory, subcontrary, subaltern, superaltern* between these classical quantifiers. The square of oppositions is used to explain relationships between propositions within traditional formal logic but we can use it also for an immediate inference. **Immediate inference** is a type of inference where we infer another quantifier from one quantifier. The square of oppositions provides a structure for how different quantifiers can be related to each other and shows what inferential conclusions we can draw if one of the statements is true or false. There are several books that deal with this topic including [4, 5, 6].

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1.1 The main goal of this paper

The goal of this paper is to present immediate inferences in Fuzzy Natural Logic (FNL). For that, we will extend the classical approach in several ways. Firstly we will add intermediate quantifiers *Almost all, Most, Many* which were introduced by Peterson [7] and their mathematical definition was introduced by Novák [8] in Łukasiewicz fuzzy type theory (Ł-FTT). Instead of the square of opposition, we will use its generalized form so-called graded Peterson's square of oppositions [9]. The truth values will be from the interval $[0, 1]$ since we use Ł-FTT. Let us note that we will deal with properties of contrary and subaltern in this paper due to the page limitation but presented ideas will work similarly for contradictory, subcontrary and superaltern.

2 Preliminaries

2.1 Mathematical bacground

Intermediate quantifiers are formally defined within a dedicated theoretical framework called T^{IQ}, which is a specialized component of Łukasiewicz Fuzzy Type Theory (Ł-FTT). In this context, we operate under the assumption that the underlying algebra of truth values is an MV_Δ-algebra, denoted as $\mathscr{E} = \langle E, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}, \Delta\rangle$ where $\mathbf{0}$ where and $\mathbf{1}$ represent the minimal and maximal elements, respectively. In most cases, we work with the standard Łukasiewicz MV_Δ-algebra, where the domain of truth degrees is given by the interval $E = [0, 1]$.

2.2 Mathematical definition of intermediate quantifiers

Below we recall mathematical definitions of fuzzy intermediate quantifiers that define graded Peterson's square of opposition. For the detail see [10].

Definition 1 *Let Ev be a formula representing an evaluative expression*[†)]*, x be variables and* A, B, z *be formulas. Then either of the formulas*

$$(Q^{\forall}_{Ev}\, x)(B, A) \equiv (\exists z)[(\forall x)((B|z)\, x \Rightarrow Ax) \wedge Ev((\mu(B))(B|z))], \quad (1)$$
$$(Q^{\exists}_{Ev}\, x)(B, A) \equiv (\exists z)[(\exists x)((B|z)x \wedge Ax) \wedge Ev((\mu(B))(B|z))]. \quad (2)$$

construes the sentence "⟨Quantifier⟩ B's are A".

Since we are limited by the number of pages, we cannot give all the examples of intermediate quantifiers in detail. Let us note that the specific examples of quantifiers are based on evaluative expressions. For example, the quantifier "Almost all" is represented by the evaluative expression "extremely big" and is marked with the letter **P**, for predominant. For further markings, we refer the reader to previous publications.

3 Properties of contrary and subaltern between quantifiers

In this section, we will focus on the study of immediate inferences which are represented by properties contrary and subaltern, which define graded Peterson's square of opposition.

[†)] The semantics of evaluative linguistic expressions is interpreted in a special formal theory T^{Ev} of Ł-FTT. (see [11]).

Proposition 2 (contrary) *Let P_1, P_2 be quantifiers such that P_1 and P_2 are contraries in the model $\mathscr{M}$. Then holds true that*

$$\mathscr{M}(P_2) \leq 1 - \mathscr{M}(P_1).$$

Proposition 3 (subaltern) *Let P_1, P_2 be quantifiers such that P_2 is subaltern of P_1 in the model $\mathscr{M}$. Then holds true that*

$$\mathscr{M}(P_1) \leq \mathscr{M}(P_2).$$

3.0.1 Demonstrative example

The idea is to use information about the truth value of the quantifier and information about the relationship of this known quantifier with other quantifiers. An example that will be dedicated to natural language claims related to **disinformation**. The most vulnerable group are seniors over 65. Below is a typical example.

- **Positive:** Almost all seniors realize that false news spreads quickly.

Below we introduce the examples of intermediate quantifiers in the particular model. Let there be a model $\mathscr{M} \models T$. Let us know that
P: "Almost all seniors realize that false news spreads quickly." *has truth value 0.9.*
Therefore, we know that **E:** "No seniors realize that false news spreads quickly" is contrary with **P**.
From that we can infer "No seniors realize that false news spreads quickly." *has truth value* $[0, 0.1]$ using Proposition 2 as follows $\mathscr{M}(\mathbf{E}) \leq 1 - 0.9$, $\mathscr{M}(\mathbf{E}) \leq 0.1$

Similarly, we can infer truth values of other quantifiers using Propositions 2-3 and using relationship of other quantifiers with a quantifier **P**. Below we introduce several examples:

- "**All** seniors realize that false news spreads quickly." *has truth value* $[0, 0.9]$. (subaltern)
- "**Most, Many, Some** seniors realize that false news spreads quickly."*all these has truth value* $[0.9, 1]$. (subaltern)
- "**No** seniors realize that false news spreads quickly.","**Almost all, Most, Many** seniors do not realize that false news spreads quickly." *all these has truth value* $[0, 0.1]$. (contrary)

We can see that for some quantifiers we obtained quite small interval of possible truth values. On the other hand for some quantifiers we obtained very wide intervals of possible truth values which do not give as much information. We are not able to directly infer truth value of quantifier **O**: Some seniors do not realize that false news spreads quickly.

4 Discussion and Future work

We showed how to use contrary and subaltern as immediate inferences in fuzzy natural logic. In general, for this inference it is necessary to know the truth of the quantifier and the relation of that known quantifier with the others. To describe the relations we have used graded Peterson's square of oppositions however there are other structures of oppositions e.g. graded hexagon of opposition [12, 13] or graded cubes of opposition [14].

In further scientific research, we will build on this publication and focus on other properties of the inference that it belongs to conversion, obversion, and contraposition.

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Large Language Models in SAT Reasoning

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1 Introduction

In this contribution, we describe the results of preliminary experiments aimed at testing whether an autoregressive Transformer can learn to imitate the steps of a sophisticated symbolic solver. Concretely, we train it to imitate the steps of a SAT solver based on *Conflict-Driven Clause Learning* (CDCL). We use our custom implementation of the solver, which contains only the most basic features.

The solver assigns Boolean values to variables in a CNF formula in a depth-first search manner (for some fixed order of variables) and backtracks whenever it finds that the current (partial) assignment is unsatisfiable. The process is implemented in a procedure we refer to as *Solve*. Inside the Solve procedure, the solver calls two other procedures: *Unit Propagation* and *Conflict Analysis*. Unit Propagation takes the current assignment and tries to deduce values for other unassigned variables. It can also detect that the current (partial) assignment is unsatisfiable (resulting in a conflict), in which case the Conflict Analysis procedure is invoked to analyze the assignment and determine the cause of the conflict. This procedure creates (learns) a new clause that prevents such a conflict from occurring again in later iterations of the solver. Adding such clauses to the set of original clauses of the CNF formula effectively restricts the search space.

We instrument the solver with logging code and record the solving traces for random CNF formulas of a certain size. The traces are divided among the three procedures in the solver (Solve, Unit Propagation, Conflict Analysis). This means that for each invocation of these procedures, we obtain one string describing the steps of the procedure, which serves as one training example for the model.

The model is trained on all traces recorded from training CNF formulas and then validated on traces from distinct validation CNF formulas.

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2 Experimental Setup

In this section, we will describe how we generate data, the chosen model architecture, and the training approach.

2.1 Data

We generate 1.2M synthetic execution traces from our CDCL solver implementation for SAT problems with 5-15 variables (in-distribution). The dataset is balanced across three trace types: 400K main solver loops, 400K unit propagation steps, and 400K conflict analysis procedures. Each trace represents a complete execution sequence of the corresponding algorithm component, including all intermediate state transitions and decisions.

The traces include oracle information blocks delimited by special "start reading" and "stop reading" tokens, which provide access to problem state during execution. These blocks contain current variable assignments, decision levels, the clause database, and reason clauses for implications. For example, an assignments read block might contain "x 1 = True, x 2 = False, x 3 = True", while clause read block provides the current formula in CNF form.

Additionally, we generate out-of-distribution test traces for problems with 16-25 variables using the same balanced composition. This OOD set tests the model's ability to generalize to larger problem instances not seen during training, which inherently require more complex reasoning to solve correctly.

We train four models: three task-specific models (one for each trace type) and one unified model trained on the mixed dataset containing all trace types. This allows us to evaluate both specialized performance as well as the model's ability to learn all three tasks at the same time.

The in-distribution and out-of-distribution datasets show substantial differences in complexity. We measured the number of solver method invocations per problem and find that for IID instances, the mean number of calls is 11.5 for Unit Propagation, 4.1 for Conflict Analysis, and 8.2 for Solve, while OOD instances show approximately 2× growth across all trace types: Unit Propagation averages 24.9 calls, Conflict Analysis 9.2 calls, and Solve 18.3 calls. This increased computational complexity is reflected in trace sequence lengths, which grow from a mean of 2,333 tokens in IID data to 3,858 tokens in OOD data.

2.2 Model

We use small-scale GPT-NeoX models [1] with 12 transformer layers, 8 attention heads, and embedding dimension of 256, resulting in approximately 9.5M parameters. We set a context window of 4096 tokens and use Rotary Position Embeddings (RoPE) [3].

Our vocabulary consists of only 88 domain-specific tokens representing CDCL algorithm operations and SAT formula elements. These include control flow tokens (`SOLVE_BEGIN`, `UNIT_PROPAGATION_BEGIN`), state markers (`READ_ASSIGNMENTS`, `READ_CLAUSES`), logical operations (`BRANCHING_VARIABLE`, `PROPAGATED`, `BACKTRACK`), and SAT-specific symbols (variable identifiers, literals, clause markers). We use a custom WordLevel tokenizer with whitespace splitting, as we specifically structure the data so that subword tokenization becomes unnecessary.

During training, we apply causal masking as in standard autoregressive models, where each token attends only to preceding tokens. Additionally, we set the loss weight to zero for two groups of tokens: (1) the initial `[BOS]` token and the following task identifier (`SOLVE_BEGIN`, `UNIT_PROPAGATION_BEGIN`, or `ANALYZE_CONFLICT_BEGIN`), and (2) all tokens within `READ_BEGIN` ... `READ_END` blocks. These oracle-provided segments (e.g., variable assignments or clause lists) remain visible to the model through causal attention but are excluded from gradient computation. Consequently, the model can condition its next actions on oracle content without being trained to reproduce it, establishing a read-only scratchpad interaction consistent with inference-time behavior (see Figure 1).

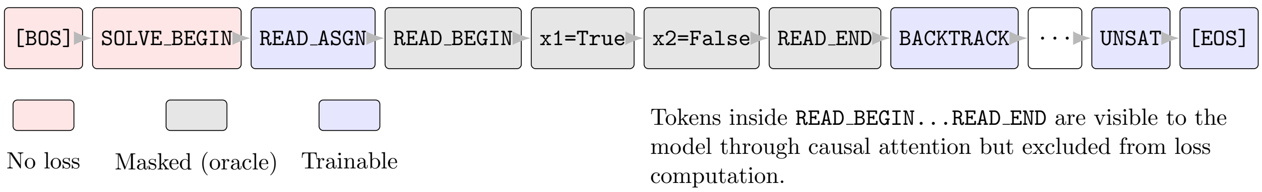

Figure 1: Training-time masking. Tokens within `READ_BEGIN...READ_END` are visible but excluded from loss.

3 Interactive Scratchpad

To enable structured reasoning over solver states, we introduce an *interactive scratchpad*—a tokenized key–value memory that the model reads and writes through explicit commands. Each key (e.g., `assignments`, `decision_levels`) stores a sequence of token IDs representing part of the CDCL state. When the model emits a `READ_<key>` command, the environment injects the corresponding content enclosed in `READ_BEGIN`...`READ_END`, making it visible through causal attention but excluded from loss computation (see Figure 1). Conversely, `WRITE_<key>` commands update or append to the stored values.

The `CDCLScratchpad` extends this interface with operations such as `level_up` and `backtrack`, which modify symbolic structures like decision levels and reason clauses by detokenizing, editing, and re-tokenizing the relevant fields.

During inference, the `AutoregressiveCDCLEnvironment` orchestrates the interaction between the model and the scratchpad. It interprets emitted tokens, injects oracle reads, stores writes, and temporarily switches contexts for subprocedures (e.g., Unit Propagation, Conflict Analysis). This setup transforms the language model into a state-aware reasoner that dynamically interacts with its symbolic environment rather than passively replaying static traces, following the broader idea of scratchpad-based reasoning introduced by Nye et al. [2].

4 Preliminary Results

We evaluate model generalization and performance using two metrics. **Token-level accuracy** measures the proportion of correctly predicted tokens at corresponding positions:

$$\mathrm{Acc}_{\mathrm{token}} = \frac{1}{N} \sum_{i=1}^{N} \frac{|\{j : p_i[j] = g_i[j]\}|}{\max(|p_i|, |g_i|)}$$

where p_i and g_i are the predicted and ground truth token sequences for example i, and N is the number of test examples. This formulation penalizes both missing and extraneous tokens.

Exact match accuracy requires complete equality:

$$\mathrm{Acc}_{\mathrm{exact}} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}[p_i = g_i]$$

Both metrics assess syntactic similarity to reference CDCL traces rather than semantic validity of the solutions. Both are computed on in-distribution (IID, 5-15 variables) and out-of-distribution (OOD, 16-25 variables) test sets. Tables 1 and 2 show the results.

Table 1: Token-level accuracy on CDCL trace generation tasks (n=512). Task-specific models are evaluated only on their corresponding task.

	Solve		**Unit Propagation**		**Conflict Analysis**	
Model	IID	OOD	IID	OOD	IID	OOD
Solve-only	0.9275	0.8740	—	—	—	—
UP-only	—	—	0.9985	0.9767	—	—
AC-only	—	—	—	—	0.9938	0.5745
Mixed	0.9999	0.9711	0.9982	0.8672	0.9962	0.7315

Table 2: Exact match accuracy on CDCL trace generation tasks (n=512). Task-specific models are evaluated only on their corresponding task.

	Solve		**Unit Propagation**		**Conflict Analysis**	
Model	IID	OOD	IID	OOD	IID	OOD
Solve-only	0.8438	0.6406	—	—	—	—
UP-only	—	—	0.9629	0.6973	—	—
AC-only	—	—	—	—	0.9531	0.4043
Mixed	0.9961	0.8770	0.9688	0.6504	0.9434	0.4707

5 Conclusion

We demonstrated that small autoregressive transformers can learn to imitate CDCL SAT solver operations with high accuracy on in-distribution problems (5-15 variables), achieving 92-99% token-level accuracy using an interactive scratchpad mechanism. While the models capture common algorithmic patterns, they struggle with the increased complexity of out-of-distribution instances with more variables and longer solving traces, suggesting that larger model capacity or improved training strategies are needed for robust generalization.

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Predicting Subgoals in Ricochet Robots with a Graph Neural Network

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1 Introduction

This contribution describes a preliminary work focused on subgoal prediction in the puzzle game called *Ricochet Robots*. It is motivated by a broader goal of designing an agent which learns to explore complex state-spaces by predicting subgoals and trying to reach them. In the ideal case, such an agent would discover useful subgoals without any supervision but in this work we focus on a much simpler task of predicting subgoals proposed by a manually designed heuristic. These tasks serve as a sanity check that confirms that a graph neural network is capable of expressing the computation required for the subgoal prediction. The text of this contribution is structured as follows: In Section 2 we describe the mechanics of the game, in Section 3 we describe the heuristic which we use to score potential subgoals, in Section 4 we describe the machine learning task and Sections 5 and 6 contain description of experimental setup and preliminary results.

2 Problem Statement

Ricochet Robots is a puzzle board game designed by A. Randolph. The game board consists of a square grid of tiles on which movable game pieces called robots of different colors and a target tile are placed. Walls may be positioned between adjacent tiles to block robot movement. Robots can move only vertically or horizontally, and once they start moving, they stop only when they hit a wall or another robot. The goal is to cover the target tile with a robot of the matching color using the minimum number of moves. In our case, we focus on games with a single target tile.

The game is NP-complete with k robots [1], making it computationally challenging.

Each move in the game consists of selecting a robot and a direction. Before executing the next move, the previous move must be fully completed. To reach the target tile, a robot may

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need assistance from other robots – these serve as stoppers, allowing the robot to halt at tiles from which it can reach the goal. Helper robots may themselves depend on assistance from other robots, making the reachability problem nontrivial. For a more detailed analysis of the game mechanics, see Masseport et al. [3].

3 A Heuristic for Scoring Subgoals

Our approach uses the hierarchical search heuristic from Hula et al. [2] for scoring potential subgoals. This heuristic decomposes the problem by identifying subgoals through graph analysis.

The game state is represented as a weighted directed graph $G = (V, E)$ where nodes are grid tiles and edges are feasible robot movements. Edges receive weights: $w(e) = 1$ for moves executable without assistance, $w(e) = 10$ for dependent moves requiring a helper robot as a blocker.

The final component $\mathcal{M}$ consists of tiles from which the target robot can reach the goal using only non-dependent edges. These are computed via reverse reachability: starting from the target tile, traverse backwards through weight-1 edges only. This identifies all positions where the target robot can complete its path independently of other robots once such a position is reached.

Transition tiles $\mathcal{S}(m)$ for each tile in the final component m are adjacent tiles outside $\mathcal{M}$ from which the target robot could approach m but requires helper assistance.

The heuristic scores each tile m in the final component $\mathcal{M}$ by computing the cost of reaching the target tile from the tile m and estimating the cost of reaching m from the current state through a tile $s \in \mathcal{S}(m)$. The estimated cost of reaching $s \in \mathcal{S}(m)$ denoted by $\text{cost}(s)$ contains the cost of reaching the tile s by the target robot but also the cost of reaching the required supporting tile by the nearest helper robot. These two costs are computed as lengths of shortest paths (from the position of the target robot to a state s and form the position of a helper robot to the respective supporting tile) which allow to cross dependent edges with $w(e) = 10$. Therefore, these costs are only estimates.

Scoring: Each target tile in the final component is scored as:

$$\text{score}(m) = \text{path\_cost}(m \to \text{target}) + \min_{s \in \mathcal{S}(m)} \{\text{cost}(s)\}$$

Tiles m in the final component with $\mathcal{S}(m) = \emptyset$ receive score ∞ (unreachable). Using this scoring, we can identify the most promising subgoals: the tile with the minimum score. The complete heuristic described by Hula et al. [2] recursively expands subgoals in the order defined by the scoring above and improves the estimates of the costs as it tries to reach the given subgoal.

4 Predicting Subgoals

Using the heuristic described above, we formulate two prediction tasks:

Task 1 (Final Component Classification): Binary classification identifying whether each node belongs to $\mathcal{M}$. This serves as a sanity check, as the task should be learnable given sufficient message passing to propagate information from the target tile backwards through non-dependent edges.

Task 2 (Best Subgoal Classification): Binary classification identifying nodes with minimum score among all tiles in the final component; the most promising subgoals the target robot should reach.

Both tasks are formulated as node-level predictions, where the model must classify every tile in the grid (node in the graph) simultaneously.

5 Experimental Setup

In this section, we will describe the task representation, model architecture, training, and evaluation.

5.1 Graph Representation

Each game instance is represented as a directed graph where nodes correspond to grid tiles and edges represent feasible robot movements. Node features are 4-dimensional binary vectors encoding: [empty, helper robot, target robot, target tile]. Edge features are 4-dimensional vectors indicating directionality and dependency: [forward non-dependent, backward non-dependent, forward dependent, backward dependent], where dependent edges require helper robot assistance.

5.2 Model Architecture

We use a recurrent Graph Attention Network (GATv2) [4] with residual connections. The architecture consists of:

- An input layer: a 4-head GATv2Conv projecting 4D node features to hidden space
- A recurrent hidden layer: single 4-head GATv2Conv applied iteratively
- An output layer: a single-head GATv2Conv followed by a task specific classifier

All layers use edge features and dropout. This recurrent design enables iterative message passing across the graph structure while maintaining parameter efficiency.

We conducted hyperparameter search varying hidden dimensions $\{64, 128\}$, recurrent iterations $[5, 10]$, learning rates $[10^{-4}, 5 \times 10^{-2}]$, dropout $[0.0, 0.5]$, and weight decay $[10^{-5}, 10^{-3}]$, with fixed batch size (128) and 4 attention heads. We report averaged performance of the top 10 models per task based on test exact match metric.

5.3 Training and Evaluation

Training uses 15,000 randomly generated game instances of 16x16 grid size, from which we use 128 instances for test and 128 instances for validation sets. We train for 100 epochs with batch size 128 using AdamW optimizer and cosine annealing with 20-epoch warmup.

We evaluate using node-level metrics (accuracy, precision, recall, F1) and graph-level **exact match ratio**: the proportion of graphs where all node predictions are correct.

6 Preliminary Results

In Task 1 (Final Component Classification), we reach near-perfect performance, where the best model obtains 100% exact match on both validation and test sets using 64 hidden channels, 7 recurrent iterations, and 0.005 learning rate. Task 2 (Best Component Classification) proves more challenging, with the best model achieving 65.6% validation and 69.5% test exact match, requiring larger capacity (128 hidden channels) and higher learning rate (0.02). Both tasks benefit from deeper recurrence (7 iterations) with 4 attention heads. The difference in model performance between tasks confirms that while identifying reachable positions is straightforward, predicting optimal subgoals requires more complex reasoning.

Table 1: Performance metrics averaged over 10 runs. Values show mean ± standard deviation.

Node-Level Metrics

Task	**Accuracy**	**Precision**	**Recall**	**F1**
Final Component	0.998 ± 0.002	0.982 ± 0.024	0.944 ± 0.069	0.961 ± 0.043
Best Component	0.998 ± 0.000	0.822 ± 0.018	0.747 ± 0.019	0.782 ± 0.013

Graph-Level Metrics (Exact Match)

Task	**Validation**	**Test**
Final Component	0.952 ± 0.042	0.961 ± 0.033
Best Component	0.663 ± 0.016	0.680 ± 0.009

This work demonstrates that graph neural networks can learn to imitate a heuristic to predict subgoals in the computationally challenging domain of Ricochet Robots, an NP-complete puzzle game. Our experiments confirm that a recurrent Graph Attention Network architecture can capture complex spatial reasoning required for subgoal identification.

We validated our architectural choices, particularly the use of recurrent message passing to propagate reachability information through the graph structure.

In future work, we plan to extend these experiments to regression tasks for direct state value prediction and compare the graph-based approach with sequence models such as autoregressive transformers.

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On the Dissimilarity of Fuzzy Information Granules

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1 Introduction

Granular Computing (GC) has evolved dynamically over recent decades, substantially enhancing methods to improve our understanding of large numerical datasets, as demonstrated, for example, by Pedrycz and Bargiela [1]. Recent developments have advanced areas such as fuzzy association rule mining and linguistic summarization, among others. Despite significant theoretical and applied achievements and numerous successful practical applications, one of the main challenges that remains is adequate validation of fuzzy information granules, and the task becomes even more complex if various granular computing approaches are confronted.

Let us consider the following two examples of fuzzy information granules:

I_1: *Almost every recording in depression has low energy.*

I_2: *If we observe low energy, then the patient is most likely suffering from depression.*

In practice, there is often a need to compare or confront different types of information granules (often imprecise) that are coming from various sources. To the best of the authors' knowledge, while there are significant contributions to particular areas (e.g., association rules mining), including approaches to assess their quality, there is not much research on the comparative analysis of different granules, which should thoroughly consider various quality criteria, types of quantifiers, and representations of linguistic expressions.

In this work, we pose the question of how to assess the dissimilarity of pairs of information granules that may be exemplified with I_1 and I_2. We focus on two representative types of information granules, namely fuzzy association rules (FAR) and fuzzy linguistic summaries, and aim to (1) propose a unified notation for the construction and selection of the most meaningful fuzzy information granules, and (2) analyze and discuss the assessment of dissimilarity across the considered types. We consider an illustrative example of real-life data collected within the mental health monitoring application scenario, which served as inspiration for this research. This work will conclude with a discussion about open challenges.

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2 Fuzzy Linguistic Summaries

Fuzzy linguistic summarization (FLS) provides natural-language descriptions of large numerical datasets and has proven to be human-consistent [2]. Originating from Yager's protoforms ('$\mathcal{Q}$ y's are $\mathcal{P}$' and '$\mathcal{Q}R$ y's are $\mathcal{P}$'), numerous approaches have since been developed. While studies (see e.g., [2, 3]) confirm their interpretability and value as information granules, clear guidelines for their practical evaluation are still lacking. This contribution builds upon previous achievements in the theories of generalized and intermediate quantifiers introduced by Mostowski [5]. We adopt the general definition of fuzzy linguistic summaries (FLS) [4] with a linguistic quantifier, qualifiers, and summarizers combined by logical connectives such as AND or OR. FLS can be exemplified with the following sentence: *Almost every sample with low energy and low variability has low speech quality.* Attributes are described by linguistic expressions represented through fuzzy membership functions that refer to relative quantities, such as *low, medium, or high.* FLS is defined as a 5-tuple (quintuple) $S = (A, L, (\mathcal{P}, \diamond), (\mathcal{R}, \star), Q)$ consisting of a set of attributes, linguistic expressions, summarizers, qualifiers, and a quantifier. For further details, we refer interested readers to [4].

The informativeness of each summary is assessed using qualitative criteria. In this work, we consider the degree of truth, expressing how accurately the summary reflects the data [6], and the degree of support, indicating how many data objects it covers. These measures are used to select the most meaningful summaries within each meta-category, as depicted in Algorithm 1.

Algorithm 1 Top-K Fuzzy Linguistic Summary (per Meta-Category)

Require: Quantifiers Q with expert ranking; feature levels; K meta-categories; measures T_1, T_2

for $i = 1$ to K **do**
  $S \leftarrow$ all summaries using features in meta-category i
  Compute $T_1(s)$ and $T_2(s)$ for all $s \in S$
  $S^\star \leftarrow \arg\max_{s \in S} T_1(s)$
  Break ties in $S^\star$ by, in order:
  (i) maximize # of summarizers,
  (ii) maximize # of quantifiers,
  (iii) prefer higher-ranked quantifiers in Q,
  (iv) maximize T_2
  Output the remaining $s^\star$
end for

3 Fuzzy association rules

Fuzzy association rules (FAR) [7] are derived from fuzzified data using quality measures such as fuzzy support, fuzzy confidence, and Łukasiewicz-based implicational quantifiers [9]. We employ the implementation from the `nuggets` R package, which extracts pairs of fuzzy antecedents and consequents, evaluates them using selected quantifiers, and filters rules with sufficiently high support and confidence. Multiple antecedents are aggregated using the minimum t-norm, chosen for its stability in high-dimensional settings where other continuous t-norms (e.g., product or Łukasiewicz) rapidly reduce activation values. To reduce redundancy, a subset-based filtration removes overly specific rules within the same consequent group, retaining only the most informative ones.

The resulting rules serve as the basis for an Implicational Model with Quantifiers (IMQ) [8], a fuzzy inference framework where each rule of the form IF A THEN B is evaluated using the Łukasiewicz implication and weighted by its corresponding quantifier. The inference process is described in Algorithm 2.

Algorithm 2 Minimal IMQ Inference from Fuzzy Association Rules

Require: Fuzzified data $\mathcal{D}$; measures (support, confidence, q); thresholds $\tau_{\text{supp}}, \tau_{\text{conf}}$
Ensure: Predicted class label(s) with ties preserved

```
R ← mine rules from D; keep if support ≥ τ_supp and confidence ≥ τ_conf
For each consequent B, remove rules whose antecedents are proper subsets of another rule's antecedents
for each instance x̄ do
    for each class C do
        Score(C) := 1
    end for
    for each rule (A ⇒ B) ∈ R do
        u ← Imp_L(A(x̄), [B = C]);    w ← q(A, B)
        Score(C) ← min(Score(C), Imp_L(w, u)) for the corresponding C
    end for
    Output arg max_y Score(C) (report all ties)
end for
```

4 Assessment of dissimilarity between information granules

Having defined the two types of granules, we now consider the challenging problem of assessing the dissimilarity score between the outcomes of FLS and FAR.

Let us denote FLS as $\mathcal{S} = \{(S_i, t_{i,1}, t_{i,2})\}_{i \in I}$, and analogously, let FAR, together with the values of the quantifier, be denoted by $\mathcal{R} = \{(R_j, q_j)\}_{j \in J}$.The list of qualitative criteria for both granules needs to be established at the beginning. For FLSs, degrees of truth $t_{\cdot,1}$ and support $t_{\cdot,2}$ are considered to assess their validity. For FARs, the degree of the quantifier q is taken into account, as it represents the degree of truth of the corresponding rule. Moreover, the degree of support s, as well as any other relevant characteristics of a rule, can be computed for each rule. Consequently, $\mathcal{R}$ can be extended to a trichotomious representation $\mathcal{R}_t = \{(R_j, q_j, s_j)\}_{j \in J}$

These requirements may serve as the foundation for defining an aggregation operator that combines individual attribute-based dissimilarities into a unified score, ensuring consistency with the trichotomous representation and monotonicity (the monotonicity of the scoring indicates that the greater the difference, the higher the value of dissimilarity).

Let us start with a small illustrative example, and let us consider a simple dissimilarity d_1 defined by

$$d_1(\mathcal{S}, \mathcal{R}) = \sum_{l \in L} |\{1 \mid (l \in \mathcal{R}) \not\equiv (l \in \mathcal{S})\}|,$$

where $L = L_{FLS} \cup L_{FAR}$, L_{FLS}, and L_{FAR} are sets of all linguistic expressions included by FLS and FAR methods, respectively. Note that in this particular example, we do not include any numerical characteristics (i.e., $\{t_{\cdot,1}, t_{\cdot,2}, q_{\cdot}, s_{\cdot}\}$); therefore, it is not necessary to specify which representation we are dealing with. Applying this formula to an illustrative example from Table 1, we receive $d_1(\mathcal{S}, \mathcal{R}) = 1$ because the only difference in linguistic expressions involved is "short sleep".

Table 1: Illustrative examples of FLSs and FAR.

No.	Fuzzy information granules
S_1	Almost every recording in depression has low voice energy.
S_2	Almost every recording in depression from a day with short sleep also has low voice energy variability.
R_1	If we observe low voice energy and low voice energy variability, then the patient is most likely suffering from depression.

In this contribution, we will investigate extensions of this definition to involve a weighting scheme based on the quality criteria, advanced handling to the missing information about the attributes, and we will attempt the semantic proximity between different linguistic expressions included in the considered two information granules (summaries and rules).

In future work, we will further extend the scoring to incorporate linguistic descriptions provided by domain experts into the comparative analyses, examining the limitations of their potential integration within the present framework, and developing advanced formulas for defining the dissimilarity score.

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Towards Responsible Time Series Forecasting

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1 Background

Ensemble techniques frequently outperform individual forecasting models, yet many common approaches remain too naive to support responsible use. Simple averaging or static weighting schemes do not account for how model performance varies across operating conditions and they also provide little insight into why a particular model should dominate the aggregated prediction at a given moment. This lack of context poses challenges in high-stakes settings such as national electricity forecasting, where both trust and accountability are essential [1]. We propose an interval Type-2 fuzzy ensembling framework that uses interpretable context descriptors to determine how forecasting models are weighted. Instead of relying on equal weighting or cumulative error metrics, the method characterizes each model's historical behavior through a collection of interval Type-2 TSK rules [2]. This produces an ensemble that not only improves accuracy but also enables a transparent connection between current operating conditions and the weights assigned to individual models.

2 Methodology

We study 24-hour-ahead electricity load forecasting. At each forecast origin, nine base models produce point forecasts. The ensemble assigns weights via fuzzy evaluation of context variables derived from the realized load.

2.1 Context Variables

Five scalar descriptors are used to characterize the system context for each day. Exogenous uncertainty is measured by the day-ahead temperature forecast error, regime change is captured by a smoothed first derivative of the load series and short-term variability is quantified through a seven-day realized volatility measure. Anomaly state is determined by a back fitting ARIMA-residual detector that flags large deviations from expected behavior, while linearity reflects the midday deviation index (MDI), describing how strongly the daily load profile strays from a a linear relationship.

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2.2 Fuzzy Characterization

Each proxy is partitioned using triangular and trapezoidal Type-1 membership functions, defined by (1) and (2) as described by Mendel [3].

$$\mu_A(x;a,b,c) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x < b, \\ \dfrac{c-x}{c-b}, & b \le x \le c, \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

$$\mu_A(x;a,b,c,d) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x < b, \\ 1, & b \le x \le c, \\ \dfrac{d-x}{d-c}, & c < x \le d, \\ 0, & \end{cases} \quad (2)$$

Membership parameters are derived from the empirical distribution using percentile-based thresholds. The anomaly indicator is represented by a two-level fuzzy set, with high membership assigned to the normal or abnormal state accordingly.

2.3 Interval Type-2 Extension via Rarity

To express epistemic uncertainty caused by atypical operating conditions, the Type-1 membership functions are extended to interval Type-2 sets. The rarity function $U(x)$ quantifies how atypical a proxy value x is with respect to its empirical training distribution and is defined as

$$U(x) = 1 - \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad (3)$$

where μ and σ denote the empirical mean and standard deviation of the proxy. Given the upper membership function $\mu_U(x)$, the corresponding lower membership $\mu_L(x)$ is obtained by

$$\mu_L(x) = \mu_U(x)\,[1 - U(x)], \quad (4)$$

which expands the footprint of uncertainty (FOU) under rare forecasting conditions.

2.4 Type-2 TSK Rule Structure

Let m be the index of the forecasting model and b the index of the fuzzy base set, for example "Weather Uncertainty High" or "Volatility Calm". Each proxy is divided into several linguistic terms, yielding multiple one-dimensional rules per model. For each model m, these rules form a dedicated interval Type-2 Takagi-Sugeno-Kang (TSK) [2] system whose output is a model-specific score. Each rule is defined as

$$R_{m,b}: \text{ IF context is in } \tilde{A}_b \text{ THEN } y = f_{m,b}, \quad (5)$$

with antecedent interval Type-2 set $\tilde{A}_b$. For a day t, this rule fires with

$$h_{L,m,b}(t) = \mu_{b,L}(c_t), \qquad h_{U,m,b}(t) = \mu_{b,U}(c_t), \quad (6)$$

where c_t is the proxy value associated with base set b. Unlike classical multidimensional TSK systems, rules operate on single-proxy inputs. Consequents are Type-1 reliability scores derived from fuzzy-weighted MAE values in each context region according to

$$f_{m,b} = -\frac{\text{MAE}_{m,b}}{\tau}, \quad (7)$$

with scaling parameter $\tau > 0$. Interval Type-2 behavior therefore results only from the antecedent firing intervals defined in (6).

2.5 Type-Reduction and Weight Computation

Type-reduction follows Karnik and Mendel's Center-of-Sets (CoS) formulation of the Karnik-Mendel algorithm [3, 10] . For forecasting model m, the family of rules $\{R_{m,b}\}_b$ yields consequents $\{f_{m,b}\}$ and interval firing strengths $\{h_{L,m,b}(t), h_{U,m,b}(t)\}$. The CoS KM algorithm produces a type-reduced interval

$$S_{m,t} = \mathrm{KM}\{f_{m,b}, h_{L,m,b}(t), h_{U,m,b}(t)\}_b, \tag{8}$$

The model weights $w_{m,t}$, which determine the contribution of each forecasting model m at time t, are obtained from the reliability scores $S_{m,t}$ via the softmax transformation, and the final ensemble prediction $\hat{y}_t^{\mathrm{TSK}}$ is computed as the weighted sum of the individual model forecasts according to (9).

$$w_{m,t} = \frac{e^{S_{m,t}}}{\sum_j e^{S_{j,t}}}, \qquad \hat{y}_t^{\mathrm{TSK}} = \sum_m w_{m,t}\, \hat{y}_{m,t}. \tag{9}$$

3 Experimental Setup

The ensemble is evaluated on Swiss hourly load data (ENTSO-E) from 1 February 2024 to 26 September 2025 [5]. Exogenous inputs include day-ahead temperature forecasts (ECMWF IFS via Open-Meteo [4]) and national holiday indicators. Nine forecasting models are used: LightGBM [7], Holt-Winters [8], TBATS, KNN-regressor, MFLES [9], NBEATSx [6], SARIMAX, seasonal naive, and MSTL. All models produce 24-hour-ahead forecasts and are retrained weekly using an expanding window. Hyperparameters are tuned via four-fold expanding-window cross-validation. Fuzzy memberships and TSK consequents are learned from data prior to 1 July 2025, and evaluation is conducted from 1 July to 26 September 2025.

4 Results

Table 1 summarizes the out-of-sample performance. LightGBM is the strongest individual model, while uniform averaging already improves upon the single-model results. The proposed interval Type-2 TSK ensemble achieves the lowest MAE and highest R^2, demonstrating small but consistent gains from context-aware weighting.

Model / Series	n	MAE	MAPE	R^2
LightGBM	13776	499	7.81	0.596
Holt–Winters	13776	522	8.15	0.569
TBATS	13776	543	8.47	0.549
KNN-Regressor	13776	552	8.54	0.531
MFLES	13776	563	8.82	0.512
NBEATSx	13776	587	9.07	0.443
SARIMAX	13776	601	9.34	0.411
Seasonal Naive	13776	614	9.55	0.369
MSTL	13776	706	10.90	0.171
Uniform Ensemble	13776	485	7.60	0.613
TSK Ensemble	13776	**480**	**7.52**	**0.624**

Table 1: Out-of-sample accuracy over the evaluation period (1 July–26 September 2025).

5 Conclusion

We introduced an interval Type-2 fuzzy ensemble for responsible time series forecasting in critical infrastructure settings. The framework combines interpretable context descriptors with rarity-based interval Type-2 antecedents, crisp TSK consequents, Center of Sets KM type reduction, and softmax aggregation. This study constitutes an initial step within a broader, ongoing research effort on context-aware and responsible ensemble forecasting, based on practical data-driven proxies that will be further refined in future work.

Applied to Swiss day-ahead load forecasting, the proposed approach outperforms all individual models and consistently improves upon uniform averaging, while providing transparent and context-aware model weighting. Future work will focus on incorporating confidence intervals and extending both model inputs and ensemble outputs to interval-valued representations, further strengthening the role of fuzzy ensembling for trustworthy and interpretable forecasting in critical domains.

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Qualitative Criteria for Fuzzy Linguistic Summaries with Absolute Linguistic Expressions

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1 Introduction

Fuzzy linguistic summarization [8] enables providing concise and easy-to-interpret descriptions about large datasets. Its main aim is to aggregate and translate numeric observations into natural-language sentences using linguistic expressions. Countless examples confirm the usefulness of fuzzy linguistic summaries in practical applications, in particular those based on sensor-collected data, when their analysis is too complex or time-consuming for experts.

This contribution builds upon previous achievements in the theories of generalized and intermediate quantifiers [1], and the evaluative linguistic expressions [2]. When constructing fuzzy linguistic summaries, one can distinguish relative expressions such as *low, high, medium, short*, etc. or absolute ones, e.g., *around 20* [9]. In the majority of the related works, the relative expressions are sufficient. However, as observed for the clinical setting, there are well-established norms arising from the medical guidelines that need to be acknowledged. Absolute expressions are of great importance in medical applications, where many indicators have numerical standards, such as blood or heart rate tests (e.g., 'About 100 patients with high blood pressure have a pulse of around 90.'). Thus, there is a need to consider absolute linguistic expressions, which are usually represented as **unimodal membership functions**, and, in our opinion, the properties of such summaries have not been studied intensely so far.

In this work, we study the antonym property of fuzzy linguistic summaries with absolute linguistic expressions. First, we briefly review qualitative evaluation criteria with a particular focus on the degree of truth (as baseline) and the degrees of imprecision and specificity. Next, we consider the property of antonym and investigate its adequacy for the selected criteria.

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2 Fuzzy linguistic summaries and their qualitative criteria

First, let us recall the definition of fuzzy linguistic summary (see [6]). For the given set of objects Y, let $A = \{A_1, \ldots, A_m\}$ denote a set of attributes, where A_i is a function $A_i\colon Y \to X_i,\ i = 1, \ldots, m$ and $X_i \neq \emptyset$. Let $L(Y, A_i)$ be a set of linguistic expressions for a given $i \in \{1, \ldots, m\}$ defined as follows $L(Y, A_i) = \{l_1^{A_i}, \ldots, l_{k_i}^{A_i}\}$ and defines the formulation of summaries in natural language. Also, let $k_i = |L(Y, A_i)|$ for given $i \in \{1, \ldots, m\}$. Moreover, let $\mathcal{V} = \{V_{i,k} \mid V_{i,k}\colon A_i(Y) \to [0,1], i = 1, \ldots, m,\ k = 1, \ldots, k_i\}$. With this notation, we have the following definition.

Definition 1 ([6]) *For Y being a set of objects, let $S = (A, L, (\mathcal{P}, \diamond), (\mathcal{R}, \star), Q)$ be a 5-tuple (quintuple) such that*

(i) $A = \{A_1, \ldots, A_m\}$ is a set of attributes $A_i\colon Y \to X_i,\ i = 1, \ldots, m$, where $X_i \neq \emptyset$,

(ii) $L = \{L_1, \ldots, L_m\}$ is the set of linguistic expressions of sets $L_i(Y, A_i) = \{l_1^{A_i}, \ldots, l_{k_i}^{A_i}\}$, $i = 1, \ldots, m$,

(iii) $\mathcal{P} \subset \mathcal{V}$ is a family of summarizers, $\mathcal{R} \subset \mathcal{V}$ is a family of qualifiers, and they satisfy

$$\exists_{i\in\{1,\ldots,m\}}\exists_{k\in\{1,\ldots,k_i\}}V_{i,k} \in \mathcal{P} \Rightarrow \forall_{k\in\{1,\ldots,k_i\}}V_{i,k} \notin \mathcal{R}$$

and

$$\exists_{i\in\{1,\ldots,m\}}\exists_{k\in\{1,\ldots,k_i\}}V_{i,k} \in \mathcal{R} \Rightarrow \forall_{k\in\{1,\ldots,k_i\}}V_{i,k} \notin \mathcal{P},$$

(iv) $Q\colon Z \to [0,1]$ is a linguistic quantifier and $Z \in \{\mathbb{R}^+, [0,1]\}$.

(v) $\diamond, \star$ are aggregation functions that can be conjunctive or disjunctive. If $card(\mathcal{P}) = 1$ or $card(\mathcal{R}) = 1$, then we assume $\diamond \equiv id$ or $\star \equiv id$, respectively.

*Then S is called a **fuzzy linguistic summary** of the following general form: $Q\ R_1 \star \ldots \star R_k$ y's are $P_1 \diamond \ldots \diamond P_l$ with $\star, \diamond \in \{AND,\ OR\}$.*

Note, that linguistic quantifiers can be seen as monadic **L**-fuzzy quantifiers of the type $< 1^n, 1 >$ [4]. For $\mathcal{P}, \mathcal{R} \subset \mathcal{V}$, we have that $\mathcal{P} = \{P_1, \ldots, P_l\}$ and $\mathcal{R} = \{R_1, \ldots, R_p\}$. For each object $y_i \in Y, i = 1, \ldots, n$, let $x_i = A_s(y_i)$ for $A_s \in A, s = 1, \ldots, m$, we take

$$R(x_i) = \begin{cases} R_1(x_i), & p = 1, \\ F_1(R_1(x_i), \ldots, R_p(x_i)), & p > 1, \end{cases} \quad \text{and} \quad P(x_i) = \begin{cases} P_1(x_i), & l = 1, \\ F_2(P_1(x_i), \ldots, P_l(x_i)), & l > 1, \end{cases}$$

where $F_1\colon [0,1]^p \to [0,1],\ F_2\colon [0,1]^l \to [0,1]$ can be conjunctions or disjunctions. Now, let $j \in \mathbb{N}$ and $\mathcal{T} = \{T_i | T_i\colon \mathcal{S} \to [0,1], i = 1, \ldots, j\}$ be the family of functions, where $\mathcal{S}$ is a family of fuzzy summaries S. $\mathcal{T}$ is called **j-tuple of qualitative criteria** of S. Here, we consider few qualitative criteria starting with the most important one - **the degree of truth** introduced by Zadeh [9]. Following that approach, we present the following formula:

$$T_1(S) = \begin{cases} Q\left(\frac{\sum_{i=1}^n C(P(x_i), R(x_i))}{\sum_{i=1}^n R(x_i)}\right), & Q \text{ is relative}, \\ Q\left(n \cdot \frac{\sum_{i=1}^n C(P(x_i), R(x_i))}{\sum_{i=1}^n R(x_i)}\right), & Q \text{ is absolute}, \end{cases} \tag{1}$$

where S is a fuzzy linguistic summary, C is a conjunctive aggregation function.
Another criteria are following (we use the above notation for FLSs):

(i) **degree of imprecision** ([5]). Let $A \in \mathcal{F}(X)$, where $\mathcal{F}(X)$ denotes a set of all fuzzy sets of $X \neq \emptyset$. The degree of imprecision of A is given by $im(A) = \frac{card\{x \in X\colon A(x)>0\}}{cardX}$. The degree of imprecision of a summarizer is given by $im_P = \sqrt[l]{\prod_{k=1}^{l} im(P_k)}$, where $P_k \in \mathcal{P},\ k = 1, \ldots, l$. Analogously, it is defined for the qualifier: $im_R = \sqrt[p]{\prod_{k=1}^{l} im(R_i)}$, where $R_k \in \mathcal{R},\ k = 1, \ldots, p$. Then the degree of imprecision of the FLS is given by

$$T_2(S) = w_R \cdot im_R + w_P \cdot im_P + w_Q \cdot im(Q), \tag{2}$$

where $w_P, w_Q, w_R \geq 0$ and $w_P + w_R + w_Q = 1$.

(ii) **the degree of specificity** ([7, 5]). Here, let us first recall a measure of specificity ([8]). A function $Sp\colon \mathcal{F}(X) \to [0,1]$ is called a measure of specificity if we have
- $Sp(A) = 1 \Leftrightarrow A = \{x\},\ \ Sp(\emptyset) = 0,$,
- $\frac{\partial Sp(A)}{\partial a_1} > 0$ and $\frac{\partial Sp(A)}{\partial a_j} \leq 0$ for all $j \geq 2$, where $A \in \mathcal{X}$ and a_j is the j-th largest membership grade in A.

Yager proposed it as follows $Sp(A) = \int_0^{\alpha_{max}} \frac{1}{card(A_\alpha)} d\alpha$, where α_{max} is the largest membership grade in A, A_α is the α-level set of A, (i.e. $A_\alpha = \{x : A(x) \geq \alpha\}$, $card(A_\alpha)$ is the number of elements in A_α. Here, we follow [5] and use $Sp(A) = \alpha_{max} -$ area under A. Now, the degree of specificity of the FLS is given by

$$T_3(S) = w_R Sp(R) + w_P Sp(P) + w_Q Sp(Q), \tag{3}$$

where $w_P, w_Q, w_R \geq 0$ and $w_P + w_R + w_Q = 1$ and Sp is the measure of specificity.

3 Examples of FLSs with absolute quantifiers

In this section, we present some examples of FLSs with absolute quantifiers and analyze the qualitative criteria for them taking into account FLSs with negations. We focus on the property of antonyms ([3]). It is actually a special case of a double negation property. However, here the only kind of negation considered for a linguistic quantifier is its antonym.

Definition 2 ([3]) *Let Q be an absolute quantifier, then its antonym quantifier is given by $Q_{Ant}(k) = Q(n-k)$, $k = 1, \ldots, n$ and n is a number of objects. The* **antonym property** *is satisfied if for a given criterion T we have*

$$T(S) = T(S_{neg}),$$

where $S_{neg} = (A, L, (\{\neg P_1, \ldots, \neg P_l\}, \diamond_D), (\mathcal{R}, \star), Q_{Ant})$ and $\diamond_D$ is a dual operator of $\diamond$.

For absolute linguistic quantifiers, we can consider their antonyms for 3 types of them: increasing, decreasing, and unimodal - which make a large part of all such quantifiers. For instance, if we know n - the number of objects, then the antonym of a quantifier 'about x' is 'about $n-x$'.

Example 3 *Let us focus on degrees of truth, imprecision, and specificity. Note that for a sentence $S =$*'About 10 employees earn low salary'*, $S_{neg} =$*'About $n-10$ employees earn **not** low salary'*, where n is the number of all employees and* '**not** low' *is a negation of a summarizer. Here, we analyze the antonym property for the following FLSs:*

(i) 'About 10 employees earn low salary.' *Here, this property is satisfied in the case of a degree of truth. For the degrees of imprecision and specificity, it is not satisfied in general. In most cases, this depends on the shape of summarizers.*

(ii) 'About 10 employees with small experience are about 30 years old.' *For this sentence, for the degree of truth, the antonym property is satisfied in some cases - it is related to the choice of the appropriate conjunction in the formula* (1). *Similarly as before, this property is not satisfied for the other two criteria.*

(iii) 'About 10 employees earn low salary and have small experience.' *Again, for the degree of truth, the antonym property is satisfied, while for the degrees of imprecision and specificity, it is not. It can be satisfied only for specific summarizers (it depends on their (non)symmetric shape with respect to the number of objects).*

In our contribution, we will analyze such FLSs with absolute expressions and comment on when the antonym property is satisfied and what it depends on.

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Few-shot learning in industrial applications

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Abstract

This paper reports on the empirical performance of few-shot learning (FSL) for visual defect classification using confidential industrial datasets. We evaluate 16 combinations of four backbone models (Perception Encoder, DINOv2, DINOv3, ConvNeXt-v2) and four FSL classifiers (Prototypical Networks, Neighborhood Component Analysis, Relation Networks, Linear Adapter). The evaluation covers three conditions: a baseline comparison, deterministic support set augmentation, and a learnable attention preprocessor. Results demonstrate that support set augmentation is a highly effective strategy, improving performance in nearly all configurations. Furthermore, the DINOv2 and ConvNeXt-V2-T backbones emerged as top performers, achieving the most competitive and highest-accuracy results, respectively. These findings suggest that for industrial FSL applications, combining a strong, pre-trained backbone with a simple augmentation strategy is a practical approach for building data-efficient classification systems.

1 Introduction

The deployment of deep learning for industrial visual quality control is often challenged by data scarcity, making it difficult to collect the large labeled datasets required for traditional supervised learning. Few-shot learning (FSL) addresses this by enabling models to generalize from very few examples. This work serves as a practical report on the application of FSL to this domain. We conduct a systematic empirical evaluation of 16 combinations of modern backbone architectures and specialized FSL classifiers on three confidential, real-world datasets for binary classification. We compare four classifiers (Prototypical Networks [5], Neighborhood Component Analysis [6], Relation Networks [7], and Linear Adapter) paired with four pre-trained vision models (Perception Encoder (PE) [1], DINOv2 [2], DINOv3 [3], ConvNeXt-v2 [4]). Our goal is to provide practical comparisons of these FSL pipelines for industrial environments.

2 Methodology

Our methodology is designed as a systematic, comparative study. We evaluate four backbone models paired with four FSL classifiers across three confidential datasets, under various experimental conditions.

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2.1 Datasets

We utilize three confidential industrial datasets for binary visual defect classification (defected/non-defected) of varying resolution, size, and difficulty. Each experiment was conducted with a balanced training set of 20 samples per class, totaling 40 samples, and a whole unbalanced test set.

2.2 Backbone Architectures

We selected four pre-trained models to serve as feature extractors, covering both Vision Transformer (ViT) and Convolutional Neural Network (CNN) architectures. All models are used with frozen weights to simulate a realistic FSL scenario where the backbone is not fine-tuned.
PE$_{CORE}$T. A Vision Transformer from the Perception Encoder family.
DINOv2 (ViT-S/14). A ViT model pre-trained with the DINOv2 self-supervised learning method.
DINOv3 (ViT-S+/16). A newer generation ViT model from the DINOv3 family.
ConvNeXt-V2-T. A modern CNN architecture, serving as a convolutional baseline.

2.3 Few-Shot Learning Classifiers

We evaluate four distinct classifiers that attach to the frozen backbones.
Prototypical Networks. A metric-learning method that computes a single prototype vector for each class as the mean of its support embeddings. Classification is performed by finding the nearest class prototype.
Neighborhood Component Analysis (NCA). An instance-based metric-learning algorithm that learns a distance metric to maximize the probability of correct classification by soft k-NN. It retains all support samples for prediction.
Relation Networks. A method that learns a deep, non-linear similarity metric. A relation module is trained to output a similarity score between query and support examples.
Linear Adapter Classifier. A fine-tuning approach where a simple linear layer, preceded by a small trainable adapter, is trained on top of the frozen backbone. This serves as a simple baseline against the other metric-learning approaches.

3 Experiments

We conducted a series of experiments to evaluate the 16 model-classifier pairs. All evaluations are performed in a 40-shot setting.

3.1 Experimental Setups

1. Baseline Evaluation. A direct comparison of all 16 model-classifier pairs to establish baseline performance on the three datasets.
2. Support Set Augmentation. We investigate the effect of expanding the support set using deterministic data augmentation. Each of the 40 support samples is augmented to create 9 additional versions, resulting in a 10x larger support set for training the classifier.
3. Attention-based Preprocessing. To address challenges with variable aspect ratios, we evaluate a learnable ‘AttentionPreprocessor‘ module. This module is inserted before the backbone and learns to generate a soft attention map, focusing on relevant image regions while avoiding aspect ratio distortion.

3.2 Evaluation Metrics

To provide a quantitative comparison, we use a set of metrics designed to evaluate both absolute performance and relative competitiveness across the three datasets. For each of the 16 model-classifier combinations, we calculate the following:

Average Raw Accuracy. The mean of the balanced test accuracy scores across the three datasets. This measures the expected absolute performance.

Average Normalized Gap. This metric measures how consistently competitive a model is. For each dataset, we first calculate a normalized gap: Normalized Gap = $\frac{\text{max\_acc}-\text{model\_acc}}{\text{max\_acc}-\text{min\_acc}}$ where 'max_acc' and 'min_acc' are the maximum and minimum accuracies achieved by any combination on that dataset. This score, which ranges from 0 (best) to 1 (worst), is then averaged across the three datasets.

Table 1: Aggregated summary of results. For each combination, 'Accuracy' and 'Norm. Gap' show the baseline value followed by the best value achieved. 'Method' indicates the experiment that yielded the best result. Select performers are highlighted.

Model-Classifier Combination	**Accuracy (%)**	**Method**	**Norm. Gap**
PE_{CORE}T - Prototypical Net.	81.15 / 86.77	Augmented	0.50 / 0.60
PE_{CORE}T - NCA	86.01 / 86.01	Baseline	0.32 / 0.32
PE_{CORE}T - Relation Net.	72.36 / 85.75	Augmented	0.75 / 0.75
PE_{CORE}T - Linear Adapter	80.98 / 83.74	Augmented	0.62 / 0.86
DINOv2 - Prototypical Net.	90.03 / 91.62	Augmented	0.09 / 0.22
DINOv2 - NCA	87.59 / 91.29	Augmented	0.20 / 0.18
DINOv2 - Relation Net.	77.88 / 92.30	Augmented	0.60 / 0.17
DINOv2 - Linear Adapter	87.98 / 89.54	Augmented	0.15 / 0.46
DINOv3 - Prototypical Net.	84.26 / 86.40	Augmented	0.33 / 0.47
DINOv3 - NCA	85.72 / 87.72	Augmented	0.25 / 0.43
DINOv3 - Relation Net.	85.19 / 86.89	Augmented	0.31 / 0.60
DINOv3 - Linear Adapter	85.51 / 88.76	Augmented	0.35 / 0.42
ConvNeXt-V2-T - Prototypical Net.	84.38 / 91.18	Augmented	0.34 / 0.51
ConvNeXt-V2-T - NCA	86.09 / 90.50	Augmented	0.27 / 0.49
ConvNeXt-V2-T - Relation Net.	87.97 / 92.92	Augmented	0.20 / 0.34
ConvNeXt-V2-T - Linear Adapter	78.51 / 89.84	Augmented	0.60 / 0.59

4 Discussion

An analysis of the results in Table 1 reveals two clear findings. First, support set augmentation is highly effective, improving performance in 15 of 16 configurations. Second, the choice of backbone is critical, with **DINOv2** and **ConvNeXt-V2-T** emerging as the top performers. The 'ConvNeXt-V2-T - Relation Net.' pairing achieved the highest accuracy (92.92%), while 'DINOv2' models delivered the most consistently competitive (lowest) normalized gaps. The experiment involving adaptive preprocessing did not yield significant improvements in this context.

5 Conclusion

This empirical study found that the most effective strategy for the evaluated industrial few-shot learning tasks was combining a modern backbone with support set augmentation. The results indicate that **DINOv2** and **ConvNeXt-V2-T** are strong feature extractors, and that augmentation is a critical step for improving performance. For practitioners, these findings suggest that focusing on these elements is a promising approach for data-efficient quality control.

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Linguistic Interpretation of Natural Data using New Forms of Intermediate Quantifiers

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Abstract

This paper examines the application of fuzzy natural logic in the analysis of scientific data and their representation through special linguistic expressions. We use the theory of evaluative linguistic expressions, which make it possible to describe quantitative data using imprecise espressions such as “very small”, “medium”, “large”, and similar. They occur in the definition of the, so called, intermediate quantifiers using which we characterize given data.

Keywords: Fuzzy natural logic, Intermediate quantifiers, Linguistic association, Graded cube of opposition

1 Introduction

Generalized quantifiers have proven effective in providing linguistic summaries of numerical data.The latter is a part of the data-to-text data-mining paradigm, which facilitates the extraction of knowledge from large numerical datasets in the form of fuzzy association rules or special summarising expressions ([1, 2]). The main aim of this paper is to build on theoretical insights into intermediate quantifiers presented in the earlier works (see [3, 4, 5]) and to propose a linguistic description of the real-world data.

Below are examples of natural language expressions with quantifiers related to graded Peterson’s cube of opposition (see [6]).

- Most people who do not eat enough vegetables are more likely to suffer from digestive problems.
- Most people who are not physically active tend to gain weight.
- Almost all people who do not get enough sleep have trouble concentrating the next day.

2 Preliminaries

2.1 Mathematical bacground

Intermediate quantifiers are formally defined within a formal theory T^{IQ} of Łukasiewicz Fuzzy Type Theory (Ł-FTT) determined by 15 special axioms. Its underlying algebra of truth values is an MV$_\Delta$-algebra $\mathscr{E} = \langle E, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}, \Delta\rangle$ where $\mathbf{0}$ and $\mathbf{1}$ are minimal and maximal elements, respectively. We usually assume the standard Łukasiewicz MV$_\Delta$-algebra, where the domain of truth degrees is given by the interval $E = [0, 1]$.

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2.2 Definition of intermediate quantifiers

Below we present mathematical definitions of fuzzy intermediate quantifiers that define the back face graded Peterson's cube of opposition. For the detail see [6].

Definition 1 (Intermediate quantifiers related to cube of opposition) *Let $Ev \in Form_{oo}$ be a formula of type oo representing an evaluative expression, $x \in Form_{\alpha}$ be a variable and $A, B, z \in Form_{\alpha}$ be formulas where α is a selected type. Then either of the formulas*

$$(Q^{\forall}_{Ev}\, x)(\neg B, \neg A) \equiv (\exists z)[(\forall x)((\neg B|z)\, x \Rightarrow \neg Ax) \wedge Ev((\mu(\neg B))(\neg B|z))], \tag{1}$$

$$(Q^{\exists}_{Ev}\, x)(\neg B, \neg A) \equiv (\exists z)[(\exists x)((\neg B|z)x \wedge \neg Ax) \wedge Ev((\mu(\neg B))(\neg B|z))]. \tag{2}$$

construes the sentence "⟨Quantifier⟩ not B's are not A".

Specific quantifier is determined by a concrete evaluative expression. For example, "Almost all" which is denoted by **P** is determined by Ev ="extremely big". For further quantifiers, see, e.g. [5].

3 Application

In this section, we present the application of the theory of intermediate quantifiers to the penguins dataset containing measurements of adult foraging penguins near Palmer Station [7], available in the R package "palmerpenguins" [8]. It contains 344 observations and 8 variables. We removed 11 observations containing missing (NA) values. We chose the following real valued variables:

- bill_length (millimeters),
- bill_depth (millimeters),
- flipper_length (millimeters),
- body_mass (grams).

We used R programming language [9] along with the "lfl" package [10]. First, we transformed values of the variables into membership degrees of fuzzy sets. For this, we applied the theory of evaluative linguistic expressions [11]. For simplicity, we considered the basic evaluative trichotomy "small", "medium", and "big" only. Then we expressed these fuzzy sets using intermediate quantifiers. Examples of the results are presented in Table 1.

Table 1: Examples of degrees of truth of various quantifiers

	Quantifier					
Implication	**all**	**almost all**	**most**	**many**	**several**	**a few**
Small bill depth ⇒ medium body mass	0.24	0.48	0.6	1	1	1
Medium bill depth ⇒ Non big flipper length	0	0.32	0.6	1	1	1
Non small flipper length⇒ big bill length	0	0	0	0.06	0.13	0.2
Non big body mass ⇒ non big bill depth	0	0.58	0.77	1	1	1

Table 1 can be used for derivation of linguistic expressions characterizing the given data. For example:

- "**Most (T)** penguins who do not have a big body mass also do not have a big bill depth." (truth value 0.77). When considering the highest truth value, we can replace the quantifier **Most (T)** by **Many (K), Several (S), A few (F)** (truth value 1).
- "**Almost all (P)** penguins who have a medium bill depth do not have a big flipper length" (truth value 0.32). Similarly, the highest truth value 1 has the quantifier **Many (K), Several (S), A few (F)**.
- "**Several (S)** penguins who do not have a small flipper length have a big bill length" (truth value 0.13).

Using the theory of logical inference, we can derive additional insight by applying selected forms of valid logical syllogisms. The following is an example of a syllogism related to the graded Peterson's cube of opposition:

$$\mathbf{dAO-IV}: \begin{array}{l} \mathscr{P}_1 : \textit{Many penguins which have not small flipper length have big bill depth.} \\ \mathscr{P}_2 : \textit{All small bill depth have medium body mass.} \\ \hline \mathscr{C} : \textit{Some penguins with small body mass have not small flipper length.} \end{array}$$

Syntactic proofs of the validity of similar syllogisms for specific cases can be found in [12]. Much richer information can be obtained when using linguistic hedges, e.g. "very, extremely, more or less, roughly", etc.

Since we are limited by the number of pages, we refer readers to the cited publications for more detailed results.

4 Conclusion

In this paper, we focused on interpreting data using generalized intermediate quantifiers related to the graded Peterson's cube of opposition. We gave examples of linguistic expressions interpreting the data. We also outlined that another related area is the theory of logical inference, which allows us to derive new conclusions.

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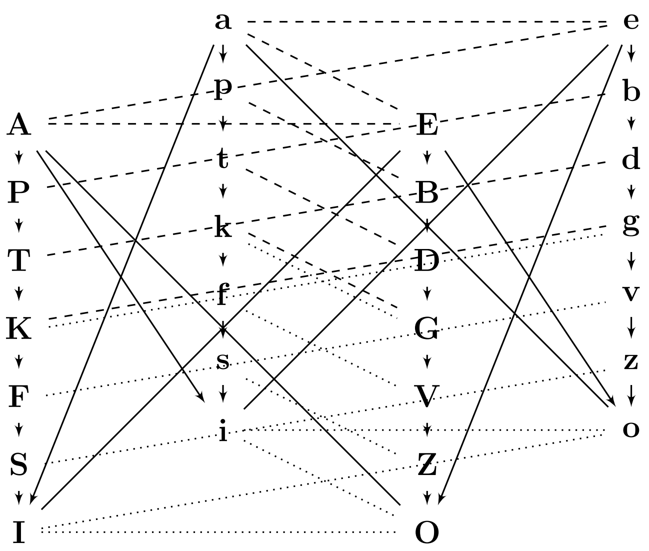

Figure 1: Graded Peterson's cube of opposition. The letters refer to specific intermediate quantifiers (see [12]).

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Verification of Validity of Syllogisms Related to Graded Peterson Cube of Opposition

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Abstract

In this article, we will examine the validity of selected forms of logical syllogisms with intermediate quantifiers. We will focus in particular on forms related to the graded Peterson's cube of opposition. Our verification will be based on the application of graded Peterson's rules using the distribution index.

Keywords: Fuzzy natural logic, Intermediate quantifiers, Distribution index, Graded cube of opposition

1 Introduction

This paper is a contribution to methods of verification of the validity of logical syllogisms with intermediate quantifiers. In previous papers, we focused mainly on formal verification of validity, i.e., we constructed a syntactic proof (see [1, 2]). Now we turn our attention to verification using generalized Peterson rules introduced in [3]. Since they are formulated in natural language, we proposed their mathematical formulation in [4]. In [5] we proved that a syllogism is valid iff all Peterson rules are satisfied. In [6] we introduced modification of these rules to syllogisms related to the graded Peterson's cube of opposition. The following are examples quantifiers from the latter.

- Most people who do not exercise regularly are at higher risk of heart disease.
- Almost all individuals who do not eat breakfast report feeling tired before noon.
- A few people who do not drink enough water suffer from reduced concentration.

Recall that many authors addressed other selected forms of logical syllogisms. Generalized syllogisms with interval quantifiers were developed in [7]. In connection with this, also extended syllogistic reasoning with the new quantifiers was proposed (see for example, [8, 9]).

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2 Preliminaries

2.1 Mathematical bacground

Intermediate quantifiers are formally defined within a formal theory T^{IQ} of (Łukasiewicz) Fuzzy Type Theory (Ł-FTT). The underlying algebra of truth values is an MV$_\Delta$-algebra $\mathscr{E} = \langle E, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}, \Delta\rangle$ where $\mathbf{0}$ and $\mathbf{1}$ are the minimal and maximal elements, respectively. In most cases, we assume that the model is the standard Łukasiewicz MV$_\Delta$-algebra whose domain of truth degrees is $E = [0, 1]$. Each formula of Ł-FTT is assigned a type $\alpha \in$ *Types* representing a certain kind of elements.

2.2 Mathematical definition of intermediate quantifiers

Definition 1 (Fuzzy intermediate quantifiers relate to cube) *Let $Ev \in Form_{oo}$ be a formula representing an evaluative expression, x_α is a variable and $A, B, z \in Form_{o\alpha}$ be formulas representing fuzzy sets. Then either of the formulas*

$$(Q^{\forall}_{Ev}\, x)(\neg B, \neg A) \equiv (\exists z)[(\forall x)((\neg B|z)\, x \Rightarrow \neg Ax) \wedge Ev((\mu(\neg B))(\neg B|z))], \quad (1)$$

$$(Q^{\exists}_{Ev}\, x)(\neg B, \neg A) \equiv (\exists z)[(\exists x)((\neg B|z)x \wedge \neg Ax) \wedge Ev((\mu(\neg B))(\neg B|z))]. \quad (2)$$

construes the sentence "⟨Quantifier⟩ not B's are not A".

For instance, the quantifier "Most" is determined by the evaluative expression *Bi Ve*, i.e., *very big*. For the details see [10].

Lemma 2 ([6]) *The following system of inequalities holds for extensions of evaluative expressions*†) *in the standard context:*

$$0 < \inf \mathrm{Supp}(\neg\, Sm\, Si) \le \inf \mathrm{Supp}(\neg\, Sm\ Ve) \le a_{\neg\, Sm\,\bar{\boldsymbol{\nu}}} = \inf \mathrm{Supp}(\neg\, Sm\,\bar{\boldsymbol{\nu}}) < 0.5$$
$$< a_{Bi\,Ve} = \inf \mathrm{Supp}(Bi\, Ve) \le a_{Bi Ex} = \inf \mathrm{Supp}(Bi Ex) \le \inf \mathrm{Supp}(Bi\, \boldsymbol{\Delta}) = 1.$$

3 Verification of selected forms of logical syllogisms

3.1 Peterson's rules

Extended Peterson's rules for verification of the validity of logical syllogisms related to the graded cube of opposition were initially proposed in [6]. Essential for them is the concept of *distribution index* (see [4, 5]).

Definition 3 (Distribution index-second face) *Let N be a set, $A, B \in \mathscr{F}(N)$.*
Let $Q^K_{Ev}(\neg B, \neg A)$, $K \in \{\forall, \exists\}$ be an intermediate quantifier determined by the evaluative expression Ev. The distribution index $\mathrm{DI}(X, Q^K_{Ev}(\neg B, \neg A))$ *(or shortly,* $\mathrm{DI}(X, Q^K)$*) of the fuzzy set $X \in \mathscr{F}(N)$ is:*

$$\mathrm{DI}(X, Q^K_{Ev}(\neg B, \neg A)) = \begin{cases} \inf \mathrm{Supp}(Ev) & \text{if } K = \forall \text{ and } X = \neg B, \\ 0 & \text{if } K = \forall \text{ and } X = \neg A, \\ 0 & \text{if } K = \exists \text{ and } X \in \{\neg A, \neg B\}, \end{cases}$$

$$\mathrm{DI}(X, Q^K_{Ev}) = \neg\, \mathrm{DI}(\neg X, Q^K_{Ev}).$$

†)By $\neg\, Sm$ and etc. we denote a group of evaluation expressions, e.g. not small, significantly small, very big, extremely big, which are constructed with the basic trichotomy small, medium, big and modified by a specific hedge. *Support* of A is a set $\mathrm{Supp}(A) = \{u \in N \mid A(u) > 0\}$. For details see Figure 1 in [11].

Formalization of Peterson's rules

1. *Rules of Distribution*

   Let $K, L \in \{\forall, \exists\}$.

(fER1) $\mathrm{DI}(X, Q^K_{\mathscr{P}_1} \oplus \mathrm{DI}(X, Q^L_{\mathscr{P}_2} = 1$ where $X \in \{M, \neg M\}$.

(fER2a) $\mathrm{DI}(X, Q^K_{\mathscr{C}}) \leq \mathrm{DI}(Y, Q^L_{\mathscr{P}_2})$ where $X, Y \in \{S, \neg S\}$.

(fER2b) $\mathrm{DI}(X, Q^K_{\mathscr{C}}) \leq \mathrm{DI}(Y, Q^L_{\mathscr{P}_1})$ where $X, Y \in \{P, \neg P\}$.

2. *Rules of Quality second face*

(fER3) Let $X, Y \in \{S, P, M\}$, $X \neq Y$ and $K \in \{\forall, \exists\}$. Then at least one of the following must hold: $\mathscr{P}_1 = Q^K_{Ev}(\neg Y, \neg X)$ or $\mathscr{P}_2 = Q^K_{Ev}(\neg Y, \neg X)$.

(fER4) Let $X, Y \in \{S, P, M\}$, $X \neq Y$ and $K, L \in \{\forall, \exists\}$. Then

$$\mathscr{C} = Q^K_{Ev}(\neg S, P) \quad \text{iff} \quad \mathscr{P} = Q^L_{Ev}(\neg Y, X)$$

where $\mathscr{P} \in \{\mathscr{P}_1, \mathscr{P}_2\}$.

3.2 Verification of validity of logical syllogisms

In this subsection, we will demonstrate application of extended Peterson's rules to validity of a logical syllogism related to the second (back) face of the graded cube of opposition. We selected a non-trivial syllogism and explain in the discussion that similar syllogisms are valid with a particular conclusion only. The following is a syllogism **pti**-III:

$\mathscr{P}_1$: Almost all $\neg M$ are $\neg P$.
$\mathscr{P}_2$: Most $\neg M$ are $\neg S$.

$\mathscr{C}$: Some $\neg S$ are $\neg P$.

Example of this syllogism from the field of wellbeing is

$\mathscr{P}_1$: Almost all people who do not get regular social support do not feel emotionally connected.
$\mathscr{P}_2$: Most people who do not get regular social support do not maintain healthy daily routines.

$\mathscr{C}$: Some people who do not maintain healthy daily routines do not feel emotionally connected.

This syllogism satisfies all the extended Peterson's rules. Indeed, from Definition 3 we know that $\mathrm{DI}(\neg M, Q^{\forall}_{Bi\,Ex}(\neg M, \neg P)) = \inf \mathrm{Supp}(Bi\ Ex_{\mathscr{P}_1})$ and $\mathrm{DI}(\neg M, Q^{\forall}_{Bi\,Ve}(\neg M, \neg S)) = \inf \mathrm{Supp}(Bi\ Ve_{\mathscr{P}_2})$. Applying of Lemma 2 we conclude that Rule (fER1) is satisfied. Furthermore, we know that $\mathrm{DI}(\neg S, Q^{\forall}_{Bi\,Ve}(\neg M, \neg S) = 0$ as well as $\mathrm{DI}(\neg S, Q^{\exists}_{Bi\,\boldsymbol{\Delta}}(\neg S, \neg P)) = 0$. It means that the Rule (fER2a) is fulfilled. Finally, $\mathrm{DI}(\neg P, Q^{\forall}_{Bi\,Ex}(\neg M, \neg P) = 0$ as well as $\mathrm{DI}(\neg P, Q^{\exists}_{Bi\,\boldsymbol{\Delta}}(\neg S, \neg P)) = 0$. It means that the Rule (fER2b) is fulfilled. Rules (fER3) and (fER4) are trivially fulfilled.

Now let us consider this syllogism with the quantifier "A few" in the conclusion. Then it does not satisfy Rule (fER2b). Indeed, using Definition 3 we obtain $\mathrm{DI}(\neg P, Q^{\forall}_{Bi\,Ex}(\neg M, \neg P)) = 0$. Furthermore, $\mathrm{DI}(\neg P, Q^{\forall}_{\neg\,Sm\,Ve}(\neg S, \neg P)) = \inf \mathrm{Supp}(\neg\ Sm\ Ve_{\mathscr{C}}) > 0$ and so, Rule (fER2b) is not satisfied.

4 Conclusion

In this paper, we have focused on further developing extended Peterson rules related to verifying the validity of logical syllogisms generated from the second face graded Peterson cube of opposition. We verified the proposed rules on a valid logical syllogism of the third figure, which is related to the area of wellbeing. In future publications, we will focus on another study that will be devoted to other selected forms of logical syllogisms.

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Measuring the Temporal Stability of Fuzzy Linguistic Summaries about Time Series with Drifts

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1 Introduction

Fuzzy linguistic summaries provide insights about large numerical datasets in natural language. While their practical potential has been demonstrated with many applications across domains, effective monitoring of evolving sequences of such summaries remains a significant challenge. This limitation is especially evident in dynamic environments such as remote health monitoring, where new data are collected continuously and human-consistent monitoring approaches are needed.

The majority of the related works concentrate on assessing the quality of individual summary sentences, typically in terms of measures such as degree of truth, confidence, support, informativeness, or focus. The evaluation of sets or sequences of summaries is considerably more complex, particularly in real-world settings where data may arrive incrementally or remain partially incomplete, thereby introducing additional uncertainty into the assessment process. Interpretability of collections of linguistic summaries has been first studied in [1] and further extended in [2].

In this work, we aim to explore the properties of the sequences of fuzzy linguistic summaries, focusing on the construction of effective and easy-to-understand information granules that support communication about changes in the observed multivariate time series. For this purpose, we adapt the fuzzy linguistic summaries based on the concept of extended protoforms [3]. Let us now briefly recall the form of fuzzy linguistic summaries (FLS) [2]:

$$S : Q\ R_1 \star \ldots \star R_k\ x \text{ are } P_1 \diamond \ldots \diamond P_l, \tag{1}$$

where $\star, \diamond \in \{\text{AND, OR}\}$, Q is a linguistic quantifier, $R_1, \ldots, R_k$ are qualifiers and $P_1, \ldots, P_l$ are summarizes. This work also builds on previous research related to the theories of generalised and intermediate quantifiers [4] initiated by the work of Mostowski [5]. Formula [[1]] can be expressed also as the generalized quantifier of type $< 1, 1 >$ [6] being an operator Q binding

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n-variables: $(Qx_, \ldots, x_n)(\varphi_1(x_, \ldots, x_n), \varphi_2(x_, \ldots, x_n))$. In this work, we consider selected examples of summaries with quantifiers *Majority* and *Around half*. The main focus of this work is on monitoring sequences of FLS. We propose a stability index and confront it with the selected well-established approaches of the statistical process control.

2 Stability assessment

Let us assume that we observe a stream of numerical data $X = x_1, \ldots, x_N$, N is large, e.g., voice features about speech collected sequentially. We assume that this stream X is divided into n segments (chunks). Fuzzy linguistic summaries (Eq.1) are constructed for consecutive segments. For further details and FLS definition, we refer to, e.g., [2].

Let $\mathcal{T}_2(S) = \{T_1^{(j)}(S), T_2^{(j)}(S)\}_{j=1}^n$ denote the sequence of degree of truth values and degree of support for a given linguistic summary S, where $T_1^{(j)}(S)$ represents degree of truth calculated for j-th segment of data. We consider the truth function (degree of truth) according to the following formula:

$$T_1(S) = \begin{cases} Q\left(\frac{\sum_{i=1}^n C(P(x_i), R(x_i))}{\sum_{i=1}^n R(x_i)}\right), & Q \text{ is relative,} \\ Q\left(n \cdot \frac{\sum_{i=1}^n C(P(x_i), R(x_i))}{\sum_{i=1}^n R(x_i)}\right), & Q \text{ is absolute,} \end{cases} \tag{2}$$

where S is a fuzzy linguistic summary, C is a conjunctive aggregation function. The degree of support is given by

$$T_2(S) = \frac{1}{n}|\{x_i : P(x_i) > 0 \wedge R(x_i) > 0\}|. \tag{3}$$

Let $\hat{T}_1^{(j)}(S)$ be its one-step predictor estimated via a modified exponentially weighted moving average:

$$\hat{T}_1^{(j)}(S) = (1-\lambda)\hat{T}_1^{(j-1)}(S) + \lambda T_1^{(j-1)}(S) + \beta d_{j-1}, \tag{4}$$

$$d_j = (1-\mu)d_{j-1} + \mu(T_1^{(t)}(S) - T_1^{(j-1)}(S)), \tag{5}$$

with initialization $\hat{T}_1^{(1)}(S) = T_1^{(1)}(S)$, $d_1 = 0$, and parameters $\lambda \in [0,1]$, $\mu \in [0,1]$, $\beta \in [0,1]$.

In previous work [7], a modified exponentially weighted moving average was considered for monitoring outcomes from the feature importance over time. In this research, we extend this idea and propose the summarization consistency index (SCI) for a fuzzy linguistic summary S to be defined as follows:

$$\mathrm{SCI}(\mathcal{T}_2(S)) = 1 - \sum_{j=2}^{n} T_2^{(j)}(S)\,\ell\left(r_S^{(j)}\right), \qquad r_S^{(j)} = \frac{T_1^{(j)}(S) - \hat{T}_1^{(j)}(S)}{s_j}, \tag{6}$$

where $T_2^{(j)}(S)$ is the degree of support calculated for fuzzy linguistic summary S on the j-th segment of data and

$$\ell(r) = \frac{r^2}{\tau^2 + r^2}, \qquad s_j = \mathrm{MAD}\left(\{T_1^{(j)}(S)\}_{j=1}^n\right) > 0, \qquad \tau > 0.$$

Intuitively, the approach captures how predictable the current degree of truth for a summary is given its historical trajectory and taking into account the corresponding degree of support. In Eq. 6, this intuition is formalized through a weighted aggregation of local deviations, where each term ℓ represents the loss function associated with the discrepancy between the observed and the expected value at time t.

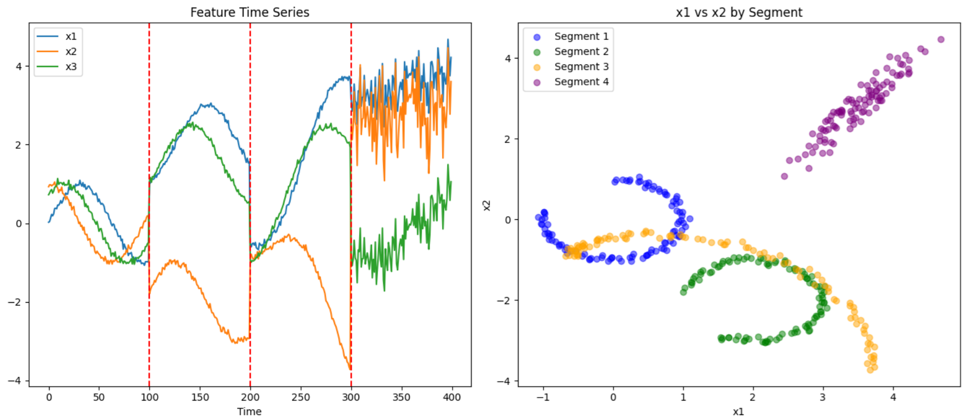

Figure 1: Simulated time series with various types of drift introduced at moment $t = 100$: sudden drift (abrupt change in mean); $t = 200$: incremental (gradual shift in mean); $t = 300$: variance change due to the increase in noise.

The SCI is bounded in $[0, 1]$, where higher values indicate smoother and more temporally consistent attributions. It is scale-invariant, achieved through robust normalization by the median absolute deviation (MAD), and adaptive to trends via exponential and momentum-based smoothing. By construction, SCI decreases monotonically with the accumulation of large, unexpected attribution shifts, making it sensitive to temporal irregularities while robust to small fluctuations.

For comparative purposes, we consider selected approaches from statistical process control. For further details, see e.g., the seminal work of Montgomery [8]. We concentrate on the methods for autocorrelated data. One approach could be that the original data are charted, but control limits are adjusted using the knowledge about the type of dependence. Alternatively, charts for monitoring residuals are constructed. To obtain the residuals, predictions are commonly calculated according to the stationary autoregressive model AR of the most appropriate order p: $X_t = \mu + a_1 X_{t-1} + \cdots + a_p X_{t-p} + Z_t$, where $X_i, i = t, t-1, \ldots$ are random observations of a process, $Z_i, i = t, t-1, \ldots$ is a series of independent random variables with constant variance and zero expectation, $a_1, \ldots, a_p$ are process parameters describing its correlation structure, and μ is a constant describing a process level.

3 Experiments

Experiments will be presented for the real-life and simulated time series with different types of introduced drift: (1) sudden (abrupt change in mean); (2) incremental (gradual shift in mean); (3) variance change (e.g., increase in noise level). Fig. 1 depicts exemplary simulated time series with these three types of drift; thus, we split the sequence into four segments of equal length. For the first segment, we consider the following three time series: $x_1(t) = \sin(0.05t) + \varepsilon_1(t)$; $x_2(t) = \cos(0.05t) + \varepsilon_2(t)$; $x_3(t) = \sin\big(0.05t + \frac{\pi}{4}\big) + \varepsilon_3(t)$ where $\varepsilon_i(t) \sim \mathcal{N}(0,\ 1)$. Then, at $t = 100$, we shift each variable by a constant (mean shift) of $c_1 = 2, c_2 = -2 and c_3 = 1.5$. Then, at $t = 200$, incremental drift is introduced. At $t = 300$, the generation mechanism changes, and the noise is particularly bigger: $x_1(t) = 0.01\,t + \eta_1(t)$; $x_2(t) = 0.8\,x_1(t) + \eta_2(t)$; $x_3(t) = \cos(0.03\,t) + \eta_3(t)$, where $\eta_i(t) \sim \mathcal{N}(0,\ 0.4^2), i = 1, 2, 3$.

Next, FLS will be constructed for the consecutive segments in natural language, and the respective quality criteria and summarization stability index will be calculated. For reference, control charts for residuals will be considered. Such control chars about two exemplary sum-

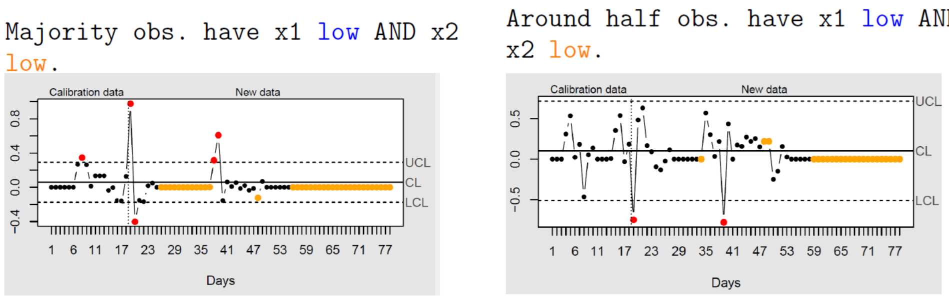

Figure 2: Two residual control charts for the linguistic summaries about simulated time series.

maries: *Majority observations have* x_1 *low and* x_2 *low.* and *Around half observations have* x_1 *low and* x_2 *low.* are illustrated in Fig. 2. As observed in this simple illustrative example, when monitoring the considered two summaries, alarms are generated in similar moments (around days 20 and 40), which correspond to the changes in the original time series generation process at $t = 100$ and $t = 200$.

In our contribution, we will thoroughly analyse the stability of sequences of FLSs with various quantifiers and various drifts, aiming to identify patterns between the statistical properties in observed multivariate time series and the calculated stability indexes. The secondary goal is to characterise the group of fuzzy linguistic summaries that may serve as promising explanations of changes detected in the original time series.

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Verification of Validity of Logical Syllogisms with New Forms of Intermediate Quantifiers Based on Grades

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Abstract

In this contribution, we continue our investigation of fuzzy Peterson syllogisms. Whereas the previous study concentrated on validating these syllogisms through the construction of formal proofs and semantic verification, the present work focuses on assessing their validity using Peterson's grade-based rules.

Keywords: Fuzzy Peterson's syllogisms, Intermediate quantifiers, Graded Peterson's square of opposition

1 Introduction

In our previous two publications, we addressed the generalized Peterson rules. First, we formally introduced their mathematical definitions based on the logical operations of Łukasiewicz algebra [1], and subsequently applied these rules to verify the validity and invalidity of logical syllogisms [2]. In the present contribution, we build upon earlier results [3], where we proposed a more general mathematical formulation of linguistically defined Peterson rules. In contrast to the previous study [1], the proposed rules do not require an explicit mathematical definition of intermediate quantifiers; instead, they rely solely on the relative position of a given quantifier within the graded Peterson square of opposition (see [4]).The novelty of this contribution is the application of other forms of intermediate quantifiers and the application of an algorithm for verifying the validity or invalidity of logical syllogisms.

2 Mathematical background

Since we are limited by the number of pages, we will only provide the most necessary mathematical definitions in order to be able to present concrete results. Therefore, we refer the reader to previous publications.

We will work with the standard Łukasiewicz-algebra.

$$\mathscr{L} = \langle [0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1 \rangle. \tag{1}$$

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2.1 Extended Peterson's rules

Below we recall the mathematical definitions of the rules of distributivity, quality and quantity [†)]. We need to know if the quantifier is *affirmative or negative.* Furthermore, we need to know a *position* of quantifier. It means to know the position inside of the graded Peterson's square [5].

A set of all considered quantifiers will be denoted by $\mathscr{Q}$. For Peterson's framework of generalized intermediate syllogisms, we have

$$\mathscr{Q} = \{\text{“all”, “almost all”, “most”, “many”, “A few”, “Several”, “some”.}\}$$

Definition 1 (Proposition) Proposition *is an expression in the form either*

$$Q(B, A) \quad or \quad Q(B, \neg A)$$

where Q is a $\langle 1, 1\rangle$ quantifier *and A, B are* formulas*. $Q(B, A)$ is an* affirmative *and $Q(B, \neg A)$ is a* negative *proposition.*

Each quantifier $Q \in \mathscr{Q}$ must have assigned a *quantity*(Q) which is explained in the subsequent definition. By *quantity*$(Q^{\text{almost all}})$ we, for example, denote the quantity of the quantifier “Almost all”.

Definition 2 (Quantity) *Let Q be a quantifier. We say that quantity(Q) fulfills the proerties as follows:*

(a) $0 < quantity(Q) \leq 1$ *for all* $Q \in \mathscr{Q}$*;*

(b) $quantity(Q_1) \leq quantity(Q_2)$ *iff for $Q_1, Q_2 \in \mathscr{Q}$, proposition $Q_2(B, A)$ is* superaltern *of proposition $Q_1(B, A)$, i.e.,* $\mathscr{M}(Q_2(B, A)) \leq \mathscr{M}(Q_1(B, A))$*;*

(c) $quantity(Q) > 0.5$ *iff for $Q \in \mathscr{Q}$, proposition $Q(B, A)$ is* contrary *to $Q(B, \neg A)$;*

(d) $quantity(Q_1) + quantity(Q_2) > 1$ *if $Q_1(B, A)$ and $Q_2(B, \neg A)$ form a* contradictory pair.

2.1.1 Grade

In Peterson's approach, the distribution index is based on the number of intermediate quantifiers. It means that the maximal value is 5 and the other values depend on the position in the Peterson's square of opposition. Our approach is based on the size that the quantifier represents in the Peterson square. In our approach, we represent this size by a **grade**, which is a value from an interval $[0, 1]$.

Definition 3 *Let us assume five basic Peterson's quantifiers. Then quantity of quantifier is represented by grade as follows:*

- $quantity(Q^{All}) = 1$, $quantity(Q^{Almost\ all}) = p$, $quantity(Q^{Most}) = t$,
- $quantity(Q^{Many}) = k$, $quantity(Q^{A\ few}) = f$, $quantity(Q^{Several}) = s$,
- $quantity(Q^{Some}) = \epsilon$

such that

[†)] Recall that all these properties were discussed in great detail in a previous publication (see [4]).

(a) $0 < \epsilon < s < f < k < 0.5 < t < p < 1 - \epsilon < 1,\ k + p > 1,\ t + k \leq 1.$

The quantity ϵ is defined as to represent the smallest quantity available so that even the antonym of $quantity(Q^{\text{Some}})$ is greater than $quantity(Q^{\text{Almost all}})$, i.e., $p < 1 - \epsilon$.

Definition 4 (Signum) *The* signum *of proposition* $\mathscr{P}$ *is defined as*

$$signum(\mathscr{P}) = \begin{cases} 1 & \text{if } \mathscr{P} \text{ is affirmative,} \\ 0 & \text{if } \mathscr{P} \text{ is negative.} \end{cases}$$

Definition 5 (Distribution) *The* distribution *of term* T *in proposition* $\mathscr{P}$, $dist(T, \mathscr{P})$, *equals to*

1. $quantity(Q)$ *if* $\mathscr{P} = Q(T, X)$;
2. ϵ *if* $\mathscr{P} = Q(X, T)$ *and* $\mathscr{P}$ *is affirmative;*
3. 1 *if* $\mathscr{P} = Q(X, T)$ *and* $\mathscr{P}$ *is negative.*

Below we introduce Extended Peterson's rules based on grades. ‡)

Definition 6 (Peterson's rules) *Let* $\mathscr{S} = \langle \mathscr{P}_1, \mathscr{P}_2, \mathscr{C} \rangle$ *be a syllogism such that* S *is the first formula of conclusion* $\mathscr{C}$, P *is the second formula of conclusion* $\mathscr{C}$ *and* M *is the middle formula.*

1. Rules of Distribution

   (R1) $dist(M, \mathscr{P}_1) \otimes dist(M, \mathscr{P}_2) > 0$;

   (R2a) $dist(S, \mathscr{C}) \leq dist(S, \mathscr{P}_2)$;

   (R2b) $dist(P, \mathscr{C}) \leq dist(P, \mathscr{P}_1)$;

2. Rules of Quality

   (R3) $signum(\mathscr{P}_1) \vee signum(\mathscr{P}_2) = 1$;

   (R4) $signum(\mathscr{P}_1) \wedge signum(\mathscr{P}_2) = signum(\mathscr{C})$.

3 Application

PPx-III:

$$\mathscr{P}_1 : \text{Almost all } \underbrace{\text{shares of companies}}_{middle formula} \text{ grow with } \underbrace{\text{growing economy}}_{predicte}.$$

$$\mathscr{P}_2 : \text{Almost all } \underbrace{\text{shares of companies}}_{middle formula} \text{ grow while } \underbrace{\text{companies are on World Stock Markets}}_{subject}.$$

$$\mathscr{C} : Q_3 \underbrace{\text{companies being on World Stock Markets}}_{subject} \text{ have } \underbrace{\text{growing economy}}_{predicate}.$$

‡) Let us recall that syllogisms can have a prescription in four figures, where the position of the subject, predicate and middle formula depends

Figure I	**Figure II**	**Figure III**	**Figure IV**
Q_1 M is P	Q_1 P is M	Q_1 M is P	Q_1 P is M
Q_2 S is M	Q_2 S is M	Q_2 M is S	Q_2 M is S
Q_3 S is P	Q_3 S is P	Q_3 S is P	Q_3 S is P

Table 1: Forms of generalized syllogism with intermediate quantifiers

	R1	R2	R3	R4	R1,R2,R3,R4
PPT	FFTF	TTFF	TTTT	TTTT	FFFF
PPF	FFTF	TTFF	TTTT	TTTT	FFFF
PPS	FFTF	TTFF	TTTT	TTTT	FFFF
PPI	FFTF	TTTT	TTTT	TTTT	FFTF

- (GR1): This rule is trivially fulfilled for all conclusions, because $dist(M, \mathscr{P}_1) = p > 0.5$ and $dist(M, \mathscr{P}_2) > p > 0.5$. Using the Definition 3 $dist(M, \mathscr{P}_1) \otimes dist(M, \mathscr{P}_2) > 0$.
- (GR2a): $dist(S, \mathscr{P}_2) = \epsilon$.
- (GR2b): $dist(P, \mathscr{P}_1) = \epsilon$. For the valid syllogism, we have to find quantifier Q_3 which has distribution at most ϵ for the subject as well as for the predicate. It is fulfilled for "Some".
- (GR3) and (GR4) are trivially fulfilled.

We can observe that the syllogism with the quantifier in both premises is valid with the quantifier "Some" in the conclusion only. We can observe that in the third figure, given the position of the subject and predicate in the antecedent, we can verify the validity of other non-trivial syllogisms with a particular conclusion in a similar way.

4 Discussion

From the Table 1 we can observe that for example the syllogism **PPT** fulfills the rule R1 in the third figure. (It is denoted by T.) In other figures the first rule is not fulfilled which is marked with the letter F. For a syllogism to be valid, it is necessary that all the rules in the given figure must be fulfilled. We can observe that only the syllogism **PPI** satisfies all the rules in the third figure, which we also proved in the analysis of the specific example above.

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Fuzzy transform for operational matrices in fractional equations

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Fractional differential and integral equations have attracted considerable attention in recent decades due to their ability to model the memory and hereditary properties of physical, biological, and engineering systems. Unlike classical (integer) derivatives, their fractional variants, such as Caputo or Riemann-Liouville derivatives, allow the system's dynamics to depend not only on its current state but also on its past behavior. As a result, fractional differential equations offer suitable models for describing anomalous diffusion and various processes with nonlocal effects. However, these models often lead to weakly singular kernels, which complicates analytical treatment and stimulates the development of various numerical approximation schemes. Below, we consider typical problems associated with fractional equations:

1. The fractional integral (Volterra) equation

$$y(t) = \mathcal{I}_{0+}^{\beta} y(t) + g(t), \tag{1}$$

2. The Initial Value Problem (IVP) with the Caputo fractional derivative

$$y(t) =_C \mathcal{D}_{0+}^{\beta} y(t) + g(t),\, y(0) = y_0, \tag{2}$$

3. The IVP with the Riemann-Liouville (R-L) fractional derivative

$$y(t) =_{RL} \mathcal{D}_{0+}^{\beta} y(t) + g(t),\, y(0) = y_0, \tag{3}$$

where y and g are the unknown and given functions, respectively. The operators $\mathcal{I}_{0+}^{\beta}$, ${}_C\mathcal{D}_{0+}^{\beta}$ and ${}_{RL}\mathcal{D}_{0+}^{\beta}$ are the fractional integral operator, the Caputo and the R-L operators of differentiation, respectively. For any suitable function y, the formal expressions of these operators are as follows:

$$\begin{aligned}
\mathcal{I}_{0+}^{\beta} y(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds, \\
{}_C\mathcal{D}_{0+}^{\beta} y(t) &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} y'(s) ds = \mathcal{I}_{0+}^{1-\beta}(y')(t), \\
{}_{RL}\mathcal{D}_{0+}^{\beta} y(t) &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left(\int_0^t (t-s)^{-\beta} y(s) ds \right) = \frac{d}{dt} \mathcal{I}_{0+}^{1-\beta} y(t),
\end{aligned} \tag{4}$$

where Γ denotes the gamma function.

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Let us introduce a function space $C^{1,1-\beta}(0,1]$, which guarantees that all equations in (4), are well-defined [2, 3] and moreover, all these equations are solvable. The space $C^{1,1-\beta}(0,1]$ consists of continuous and continuously differentiable functions f in $(0,1]$ such that the following inequalities are valid:

$$\begin{cases} |f(t)| \le b, \\ |f'(t)| \le bt^{\beta-1}. \end{cases}$$

It is known that $C^{1,1-\beta}(0,1]$ is a Banach space where the norm is given by

$$\|f\|_{C^{1,1-\beta}(0,1]} = \sup_{t\in(0,1]} |f(t)| + \sup_{t\in(0,1]} t^{1-\beta}|f'(t)|.$$

It has been shown in [2] that the considered operators $\mathcal{I}^{\beta}_{0+}$, ${}_{C}\mathcal{D}^{\beta}_{0+}$, ${}_{RL}\mathcal{D}^{\beta}_{0+}$ are compact on $C^{1,1-\beta}(0,1]$. This fact guarantees that the approximation can be found in the form of a member of a convergent (to a solution) sequence.

We assume that $g \in C^{1,1-\beta}(0,1]$, so that the equations (1), (2), and (3) have unique solutions [2, 3]. However, in most cases, the solutions cannot be represented analytically. Therefore, approximation methods are required. With this idea in mind, we propose the application of the F-transformation methodology [5, 6, 7], which can reduce the original problems of (1), (2), and (3) to the corresponding systems of linear equations.

In this paper, we show that after applying the F-transform to the problems (1), (2) and (3), we obtain three systems of linear equations with the same matrix, which is either in direct or inverse form. We call this matrix as *operational matrix.*

First, let us give the necessary brief information about the F-transform method [4, 5].

Definition 1 (Fuzzy partition) *Let $n \ge 3$. Fuzzy sets $A_0, A_1 \cdots, A_n : [0,1] \to [0,1]$, identified with their continuous membership functions, constitute a fuzzy partition of $[0,1]$ with nodes $0 = t_0 < t_1 < \cdots < t_n = 1$, if they fulfill the following conditions:*

1. *(locality) For all $k = 1, \cdots, n-1$, $A_k(t) = 0$ if $t \notin (t_{k-1}, t_{k+1})$; $A_0(t) = 0$ if $t \notin [0, t_1)$; $A_n(t) = 0$ if $t \notin (t_n, 1]$;*
2. *(positiveness) For all $k = 1, \cdots, n-1$, $A_k(t) > 0$ if $t \in (t_{k-1}, t_{k+1})$, $A_0(t) > 0$ if $t \in [0, t_1)$; $A_n(t) > 0$ if $t \in (t_{n-1}, t_n]$;*
3. *(normality) For all $k = 0, \cdots, n$, $A_k(t_k) = 1$.*

Remark 1 *Let $\gamma \ge 1$. Fuzzy partition of $[0,1]$ with nodes $0 = t_0 < t_1 < \cdots < t_n = 1$, is said to be* graded *if*

$$t_i = \left(\frac{i}{n}\right)^{\gamma}, \; i = 0, \cdots, n. \tag{5}$$

Definition 2 (Direct and Inverse F^0-transform) *Let $n \ge 2$, and $A_0, \ldots, A_n$ be a fuzzy partition of $[0,1]$ with nodes $t_0, t_1, \ldots, t_n$. Let function $f \in L^2[0,1]$. Then,*

- *the (direct) F^0-transform of f is a vector of real numbers $(F^0_0, \ldots, F^0_n)$ such that*

$$F^0_k = \frac{\int_{t_{k-1}}^{t_{k+1}} f(t)A_k(t)dt}{\int_{t_{k-1}}^{t_{k+1}} A_k(t)dt},$$

where we set $t_{-1} = t_0 = 0$ and $t_{n+1} = t_n = 1$;

- *the inverse* F^0*-transform of* f *is a function* $\hat{f}^0_F \in C[0,1]$ *represented as*

$$\hat{f}^0_F(t) = \sum_{k=0}^{n} F^0_k A_k(t),$$

where $(F^0_0, \dots, F^0_n)$ *is the direct* F^0*-transform of* f*.*

Remark 2 *The direct* F^0*-transform of a function* $f \in L^2[0,1]$ *can be considered as result of a linear operator* $\mathbf{F}^0 : L^2[0,1] \to \mathbb{R}^{n+1}$ *so that*

$$\mathbf{F}^0[f] = (F^0_0, \dots, F^0_n).$$

Our goal is to apply the operator $\mathbf{F}^0$ to both sides of the equations (1), (2) and (3) (whose solutions are denoted by y) and transform each of them into a corresponding system of linear equations with respect to the unknown vector $\mathbf{F}^0[y]$, which is the direct F^0-transform of the function y.

In particular, the equation (1) will be rewritten as

$$\mathbf{F}^0[y] = \mathbf{F}^0[\mathcal{I}^\beta_{0+} y] + \mathbf{F}^0[g], \tag{6}$$

where $\mathbf{F}^0[y]$ and $\mathbf{F}^0[g]$ are vectors of F^0-transform components of y and g. To transform the equation (6) into a system of linear equations, we need to show that the vector $\mathbf{F}^0[\mathcal{I}^\beta_{0+} y]$ can be represented as $L\mathbf{F}^0[y]$, where L is the corresponding operational matrix that approximates the linear transformation of y performed by the integral operator $\mathcal{I}^\beta_{0+}$.

Further details will be given for the equation (6). Assume that $y \in C^{1,1-\beta}(0,1]$ is the exact solution of equation (1), and $0 < \beta < 1$. Assume also that all three functions from (1) are smooth and their second derivatives are bounded in the interval $(0,1)$. For the sake of simplicity we assume that limit initial values are: $y(0) = g(0) = 0$.

Choose $0 < \varepsilon < 1$, $n = \left\lceil \frac{1}{\varepsilon} \right\rceil$, $\gamma \geq \frac{1}{\beta}$, and arrange the nodes $t_1, \dots, t_n$ in $[0,1]$ according to (5). Suppose that the fuzzy partition of $[0,1]$ consists of the triangular-shaped fuzzy sets $A_0, A_1, \dots, A_n$ that fulfill the so called Ruspini condition (see [4] for the details).

Since zero initial value we may assume that $\mathbf{F}^0_0[y] = \mathbf{F}^0_0[g] = 0$. Therefore, the equation (6) can be considered with the reduced to n dimension with respect to the unknown F^0-transform components of y, which we denote as $\mathbf{Y} = (\mathbf{Y_1}, \dots, \mathbf{Y_n})$. Similarly the reduced vector of F^0-transform components of g will be denoted as $\mathbf{G} = (\mathbf{G_1}, \dots, \mathbf{G_n})$.

Theorem 3 *Let the assumptions given above be fulfilled. Then the* n*-dimensional vector of* F^0*-transform components* $\mathbf{F}^0[\mathcal{I}^\beta_{0+} y]$ *can be represented as* $\underline{L}^\beta \mathbf{Y}$*, where* $\underline{L}^\beta$ *is the operational matrix, corresponding to the integral operator* $\mathcal{I}^\beta_{0+}$*, and*

$$\underline{L}^\beta = \frac{1}{\Gamma(\beta)} \begin{pmatrix} a^\beta_{1,1} & 0 & \cdots & 0 \\ a^\beta_{2,1} & a^\beta_{2,2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a^\beta_{n,1} & a^\beta_{n,2} & \cdots & a^\beta_{n,n} \end{pmatrix},$$ *where we use the following notation:*

- $\Lambda_{k,l} = k - l$, $k, l \in \mathbb{N}$*;*
- *for* $k = 1, \dots, n$ *and* $i = 1, \dots, k$,

$$a^\beta_{k,i} = \begin{cases} \frac{1}{n^\beta \beta(\beta+1)} \left(\frac{(\Lambda_{k,i+1})^{\beta+1} - (\Lambda_{k,i})^{\beta+1}}{\Lambda_{i+1,i}} + \frac{(\Lambda_{k,i-1})^{\beta+1} - (\Lambda_{k,i})^{\beta+1}}{\Lambda_{i,i-1}} \right), & \text{if } 1 \leq i \leq k-1, \\ \frac{1}{n^\beta \beta(\beta+1)}, & \text{if } i = k. \end{cases}$$

The Euclidean distance $\|\mathbf{F}^0[\mathcal{I}^{\beta}_{0+}y] - \underline{L}^{\beta}\mathbf{Y}\| < \varepsilon$.

Remark 3 *It is easy to see that the matrix* $\underline{L}^{\beta}$ *is not singular.*

Corollary 4 *Let the conditions of the theorem be satisfied. Then the equation (6) can be approximately rewritten as:*

$$\mathbf{Y} = \underline{L}^{\beta}\mathbf{Y} + \mathbf{G},$$

so that the direct F^0*-transform* $\mathbf{Y}$ *of the exact solution* y *of equation (1) can be found using*

$$\mathbf{Y} = (E - \underline{L}^{\beta})^{-1}\mathbf{G}.$$

The approximate solution of equation (1) can be represented as the inverse F^0*-transform, that is:*

$$\hat{y}^0_F(t) = \sum_{k=1}^{n} Y^0_k A_k(t).$$

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Evaluation of Machine-Learning Models in Polymer Chemistry with Prediction of Not Reported Measurements

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We present a data-centric evaluation framework tailored for heterogeneous, unevenly distributed scientific datasets, using polymer nanoparticles as a case study. This work aims to extend the application of machine learning (ML) within the chemical sciences. In many chemistry-related problems, the primary challenge lies not in the selection of algorithms but in the evaluation process, particularly when datasets are limited in size, comprise mixed data types (categorical and numerical) and exhibit uneven data distribution. ML enables the modeling of these dependencies and guides experiments through data-driven predictions. We evaluate multiple ML models to identify the one that most effectively captures the underlying patterns in polymer data and yields accurate predictions of polymer characteristics, as measured by low mean absolute error (MAE). The selected model is then used to identify key features that influence the target properties and to impute missing ("Not Reported") values, thus completing the dataset for downstream analyses. Our findings regarding algorithm behavior are specific to the configurations evaluated, whereas the evaluation framework can be applied to other datasets with similar characteristics beyond polymer chemistry.

1 Introduction

Machine Learning offers a robust tool for capturing complex data relationships and directing experimental efforts through predictive insights. Polymeric nanoparticle performance results from the interaction of multiple factors, including polymer architecture, composition, and processing conditions. Traditional experimental testing is costly and time-consuming, especially when measurements are incomplete or vary widely in scale. By learning patterns across chemical datasets, ML accelerates discovery, reduces cost, and reveals mechanistic signals that are hard to see by experiment alone.

In their survey, Ge et al. [1] report that ML helps connect polymer structures to their properties, but progress is often slowed by small, messy, and inconsistently prepared datasets. To make real use of ML, chemical data need to be translated into clear, machine-readable descriptors and follow FAIR principles—findable, accessible, interoperable, and reusable [6]. The field grows faster when chemists and data scientists work together to refine models for specific polymer systems, improve data quality, and apply ML to study solid-state behavior, solution properties, composites, drug delivery, and polymer–biology interactions. Prior works [3, 5] have improved the accuracy of chemical property, but reproducibility in limited sample and mixed

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Model	**Fold**										**Avg Test**
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**MAE**
XGB_{Confg}	3.90	28.42	8.91	29.66	11.69	17.20	96.57	58.78	13.28	114.93	38.34 ±36.99
MLP_{Confg}	64.18	43.03	13.29	71.23	34.93	53.15	124.91	117.38	50.49	78.70	65.13 ±33.12

Table 1: Fold-wise test MAE (10 folds and mean±SD) for XGB_{Confg} and MLP_{Confg} on EM Nanoparticle Size(nm)

type data still depends on careful evaluation. It requires checking how well prediction errors generalize across folds, identifying the key descriptors that influence polymer behavior, and accurately predicting missing features. Building on these principles, our study delivers a compact, reproducible pipeline and demonstrates its benefits on evaluation of polymeric characterized properties while maintaining a balanced chemical and computational perspective.

To address this setting, we adopt a data-centric ML workflow that balances chemical interpretability with statistical robustness. We curate the diblock copolymer dataset defining descriptors and based on those descriptors their polymer sizes. We benchmark five regressors spanning linear, tree/ensemble, gradient-boosted, and neural models under consistent settings with k-fold cross validation to ensure fair comparison [2]. We use model-based attribution to highlight key features, and finally predict the "Not Reported" measurements to advance later-stage analysis.

2 Methodology

Polymer nanoparticle dataset is curated though surveyed publications up to 2024, containing 118 diblock copolymer samples with 13 descriptors as features/attributes (chemical composition, core/corona DP, core cLogP; concentration(wt%), characterization pH, LCST/UCST, and related conditions) and two particle-size outputs, namely, EM nanoparticle size and Hydrodynamic diameter. We repeatedly cleaned and adjusted the dataset in several meetings to make the chemistry data suitable for ML, identify problematic samples, and improve the results until the predictions became reliable. To obtain unbiased performance estimates under heterogeneous and uneven data distribution constraints, we employed 10-fold cross-validation. We benchmarked five regression models with complementary inductive biases: Ridge and Lasso regression (both with default parameters), Random Forest (`n_estimators`=500, `random_state`=42), XGBoost (`n_estimators`=500, `learning_rate`=0.05, `max_depth`=4, `subsample`=0.8, `colsample_bytree`=0.8, `random_state`=42), and a Multilaye Perceptron (MLP) as an artificial-neural-network baseline(`hidden_layer_sizes`=(20,10), `max_iter`=600, `early_stopping`=True, `random_state`=42). XGB_{Confg} and MLP_{Confg} represent the tested XGB configuration and tested MLP configuration, respectively. We report test MAE for each fold and summarize uncertainty as the mean ± SD across folds to compare the performance of each regressor. Predicted-versus-observed plots for models (one per fold) were used to visualize fit quality. For each plot, we record the input-feature combinations of these outliers to assess whether specific chemical compositions or conditions systematically challenge the regressor. Also, using fold-wise error spread, we select the most stable model and examine which descriptors are most important for the nanoparticle sizes. Then we refit the model on all labeled data to estimate previously unreported measurements.

Model	Fold										Avg Test MAE
	1	2	3	4	5	6	7	8	9	10	
XGB_{Confg}	28.79	32.53	41.12	5.54	22.25	99.78	18.78	40.14	25.88	73.37	38.92 ±26.70
MLP_{Confg}	29.62	66.51	45.84	80.38	55.04	172.17	18.94	36.85	25.88	18.65	54.98 ±43.62

Table 2: Fold-wise test MAE (10 folds and mean±SD) for XGB_{Confg} and MLP_{Confg} on Hydrodynamic Diameter(nm)

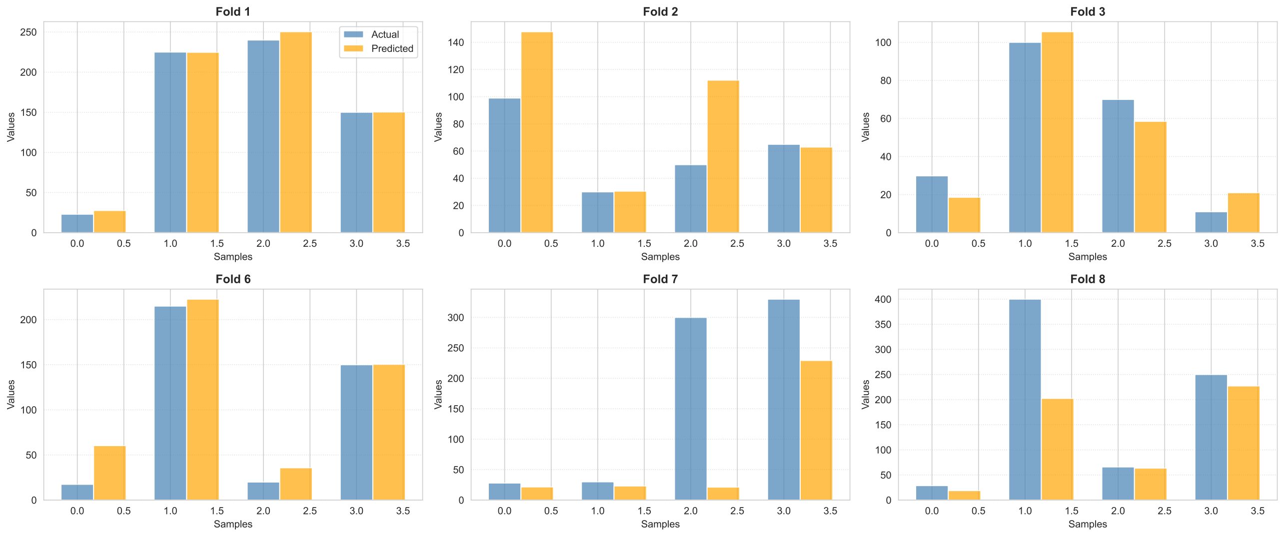

Figure 1: Actual vs Predicted Test Values (Folds 1-3 and 6-8) with XGB_{Confg} on EM Nanoparticle Size(nm)

3 Results and Discussion

We compute train and test set MAE's for each regressors. With the limiliting space, we report only a subset of results. Tables 1–2 report test MAE per target for XGB_{Confg} and MLP_{Confg}, showing that the tree-based boosting model XGB_{Confg} outperforms the neural-network model MLP_{Confg} on this polymeric dataset. Figure 1 shows predicted-vs-observed plots under 10-fold cross-validation for the top performer XGB_{Confg} model on electron microscopy (EM) nanoparticle size for folds 1-3 and 6-8. These fold-wise panels indicate that most folds fit well, while sparse regimes in certain folds degrade performance and inflate the average MAE. From these fold-wise plots, we flag individual samples whose predicted diameters deviate substantially from their true values.

In Figure 2, we present XGB_{Confg}'s gain-based feature attribution to identify the most influential descriptors. For each split, the algorithm computes the reduction in loss achieved by splitting on a feature and averages these gains across all trees. Features with higher average gain are interpreted as more influential on the predicted output. For the size of EM nanoparticle, 'temperature' is most impactful attribute, followed by 'characterization pH', 'corona DP', 'LCST', and 'concentration (wt%)'. For Hydrodynamic diameter, the most influential attributes are 'core cLogP', 'temperature', 'concentration (wt%)', 'end-group charge', and 'core DP'. Afterwards, to complete the record, we refit the configured XGB_{Confg} model on all labeled samples and generated predictions for entries with "Not Reported" polymeric sizes.

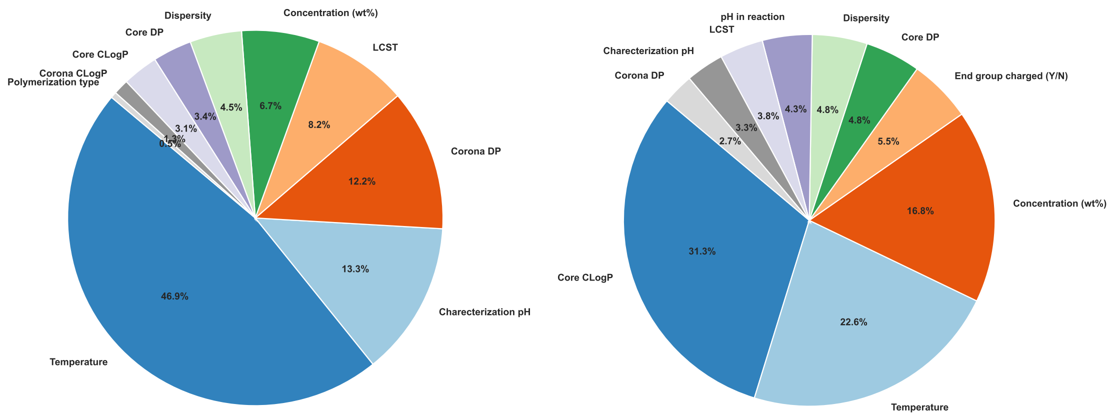

Figure 2: XGB_{Confg} Gain-based Feature Attribution for (left) EM Nanoparticle Size(nm), (right) Hydrodynamic Diameter(nm)

4 Conclusion

This work showcases the application of machine learning to polymer–nanoparticle data. Using 10-fold cross-validation, we evaluated the generalization error of multiple regression models under identical preprocessing pipelines and fixed hyperparameter settings. In our experiments, the tested configuration of XGB_{Confg} achieved the lowest test MAE and the most consistent performance across folds, whereas the tested neural baseline MLP_{Confg} showed higher fold-to-fold variance. We do not claim superiority of entire method families (e.g., all boosting or all ANN); results are specific to this dataset and configurations. Furthermore, model-based feature attribution identified the most influential predictors of polymer size. Finally, after selecting the most stable regressor, we retrained it on the full labeled dataset and successfully estimated previously unreported polymer particle-size values. Overall, we present a straightforward and reproducible ML pipeline for interpretable chemical modeling in settings where data are limited and unevenly distributed.

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Relation of Complete Correlation and Its Implication on Interval Operations

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Abstract

This paper lays the groundwork for defining division and multiplication on fuzzy intervals under complete correlation. We introduce joint possibility distributions to model dependencies between fuzzy variables with complete correlation expressed via a linear relation. We then examine its impact on interval operations and the inverse property. Our results show that fuzzy arithmetic requires more nuanced approaches than simple point-wise interval analogies.

1 Preliminaries

We begin by introducing fuzzy intervals (or fuzzy numbers) and their α-cuts, which generalize real intervals and are essential for later definitions. A fuzzy set A on $\mathbb{R}$ is called a **fuzzy interval** if it is normal, fuzzy convex, upper semi-continuous, and has bounded support [1]. Its **α-cuts** are defined as $[A]_\alpha = \{x \in \mathbb{R} \mid A(x) \geq \alpha\}$ for $\alpha \in (0, 1]$, and each is a closed interval $[a_\alpha^-, a_\alpha^+]$. The set of all fuzzy numbers is denoted by $\mathbb{R}_\mathcal{F}$. A fuzzy interval is **symmetric** about $x \in \mathbb{R}$ if $A(x - y) = A(x + y)$ for all y; otherwise, it is **non-symmetric**. Next, we introduce complete correlation via joint possibility distributions and the sup-J extension principle, following [2, 3].

Definition 1 *An **n-dimensional possibility distribution** on $\mathbb{R}^n$ is a fuzzy subset J with a membership function $J\colon \mathbb{R}^n \to [0, 1]$ satisfying $J(x_0) = 1$ for some $x_0 \in \mathbb{R}^n$.*

The family of n-dimensional possibility distributions will be denoted $\mathcal{F}(\mathbb{R}^n)$.

Definition 2 *Let $A_1, \ldots, A_n \in \mathbb{R}_\mathcal{F}$ be fuzzy intervals, then $J \in \mathcal{F}(\mathbb{R}^n)$ is called a **joint possibility distribution** if for each $i = 1, \ldots, n$, the following condition holds:*

$$\bigvee_{x_j \in \mathbb{R}, j \neq i} J(x_1, \ldots, x_n) = A_i(x_i), \quad \forall x_i \in \mathbb{R}.$$

Each A_i is called the i-th marginal possibility distribution of J.

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Definition 3 *Fuzzy intervals A, B are said to be* ***completely correlated*** *if there exist $q, r \in \mathbb{R}$, $q \neq 0$ such that their joint possibility distribution J_c is given by*

$$J_c(x,y) = A(x)\chi_{\{qx+r=y\}}(x,y) = B(y)\chi_{\{qx+r=y\}}(x,y), \tag{1}$$

where $\chi_{\{qx+r=y\}}$ stands for the characteristic function of the line $\{(x,y) \in \mathbb{R} \mid qx + r = y\}$ and $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$, $[B]_\alpha = q[A]_\alpha + r$, $\alpha \in [0,1]$.

Remark 4 *Fuzzy intervals A and B are said to be completely* ***positively (negatively)*** *correlated if q is positive (negative).*

Remark 5 *Symmetric fuzzy intervals can be both positively and negatively correlated at the same time.*

Definition 6 *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let $J \in \mathcal{F}(\mathbb{R}^2)$ be a joint possibility distribution of $A, B \in \mathbb{R}_\mathcal{F}$. The* ***sup-J extension*** *of f to a fuzzy function $f_J\colon \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(\mathbb{R})$ is defined by*

$$f_J(A,B)(y) = \bigvee_{y=f(x,y)} J(x,y).$$

Theorem 7 *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let $J \in \mathcal{F}(\mathbb{R}^2)$ be a joint possibility distribution of $A, B \in \mathbb{R}_\mathcal{F}$. Then $[f_J(A,B)]_\alpha = f([J]_\alpha)$, $\forall \alpha \in [0,1]$. If $f_J(A,B)$ is a fuzzy number, then*

$$[f_J]_\alpha = \left[\bigwedge_{(x,y)\in[J]_\alpha} f(x,y), \bigvee_{(x,y)\in[J]_\alpha} f(x,y) \right].$$

2 Relation between Division and Multiplication of Completely Correlated Intervals

In this section, we define the operations division and multiplication of two real intervals which are connected by the complete correlation given by Definition 3. Recall that real intervals can be interpreted as special cases of α-cuts of fuzzy intervals. In addition, we present the invertibility property between these two operations.

Definition 8 *[4] Let $A = [a^-, a^+]$, $B = [b^-, b^+]$, $0 \notin A$, be real compact intervals and let $J_c \in \mathcal{F}(\mathbb{R}^2)$ be the complete correlation relation of A, B defined by (1). The J_c**-division** is the real interval $B \div_{J_c} A$ defined by*

$$B \div_{J_c} A = \begin{cases} \left[q + \frac{r}{a^-}, q + \frac{r}{a^+}\right], & r < 0, \\ \\ \left[q + \frac{r}{a^+}, q + \frac{r}{a^-}\right], & r \geq 0. \end{cases} \tag{2}$$

Definition 9 *Let $A = [a^-, a^+]$, $B = [b^-, b^+]$ be real intervals, and let $J_c \in \mathcal{F}(\mathbb{R}^2)$ be the complete correlation relation of A, B defined by (1). The J_c**-multiplication** of A, B is the real interval $A \odot_{J_c} B$ defined by*

$$A \odot_{J_c} B = [\min S, \max S]^* \quad \textit{where} \quad S = \{a^-(qa^- + r), a^+(qa^+ + r)\}, \tag{3}$$

*and * defines the following exceptions:*

1. $A > 0, B > 0$, $q < 0$ *with* $r \in (-qa^+, -2qa^+)$ *or* $A < 0, B < 0$, $q < 0$ *with* $r \in (-2qa^-, -qa^-)$*:*

$$A \odot_{J_c} B = \left[\min S,\ \bigvee_{a\in A} a(aq+r)\right].$$

2. $A < 0, B > 0$, $q > 0$ *with* $r \in (-qa^-, -2qa^-)$ *or* $A > 0, 0 \in B$, $q > 0$ *with* $r \in [-qa^+, -2qa^-)$

$$A \odot_{J_c} B = \left[\bigwedge_{a\in A} a(aq+r),\ \max S\right].$$

3. *Analogous exceptions for cases* $A > 0, 0 \in B$ *or* $A < 0, 0 \in B$.

Remark 10 *Based on Remark 5, real intervals are positively and negatively correlated with respect to the parameter* q*. Therefore, the results of division or multiplication of such correlated intervals differ according to the particular correlation and parameter* q*.*

Example 11 *Consider the intervals* $A = [3, 6]$ *and* $B = [6, 15]$*.*

1. *A, B are positively correlated with* $q = 3$*,* $r = -3$*.*
   Then $B \div_{J_c} A = \left[2, \frac{5}{2}\right]$ *and* $A \odot_{J_c} B = [18, 90]$*.*

2. *A, B are negatively correlated with* $q = -3$*,* $r = 24$*.*
   Then $B \div_{J_c} A = [1, 5]$ *and* $A \odot_{J_c} B = \left[36,\ \bigvee_{a\in A} a(-3a+24)\right]$*.*

Proposition 12 *For the following combinations:*

1. $q > 0$ *and* $A > 0, B > 0$ *or* $A < 0, B < 0$*,*

2. $q < 0$ *and* $A > 0, B < 0$ *or* $A < 0, B > 0$*,*

the bounds of J_c*-multiplication coincide with the standard interval multiplication.*

Theorem 13 *Let* $A = [a^-, a^+]$*,* $B = [b^-, b^+]$ *be real intervals, and let* $J_c \in \mathcal{F}(\mathbb{R}^2)$ *be the complete correlation relation of* A, B *defined by (1). Let* $C = [c^-, c^+]$ *be the real interval such that* $C = B \div_{J_c} A$ *is given by (2). Then there exists* $J_{c'} \in \mathcal{F}(\mathbb{R}^2)$ *of* A, C *given by (1) with parameters* q' *and* r' *such that the following holds*

$$C = B \div_{J_c} A \Rightarrow B = A \odot_{J_{c'}} C.$$

Proof Assume $A = [a^-, a^+]$, $B = qA + r = [\min\{qa^- + r, qa^+ + r\}, \max\{qa^- + r, qa^+ + r\}]$, and $C = B \div_{J_c} A$. By the definition of J_c-division, $C = \left[\min\{q + \frac{r}{a^-}, q + \frac{r}{a^+}\}, \max\{q + \frac{r}{a^-}, q + \frac{r}{a^+}\}\right]$. Based on the structures of A, B and C, we can always define a complete correlation $J_{c'}$ between A and C such that $A \odot_{J_c'} C = B$. ■

Theorem 14 *Let* $A = [a^-, a^+]$*,* $B = [b^-, b^+]$ *be real intervals, and let* $J_c \in \mathcal{F}(\mathbb{R}^2)$ *be the complete correlation relation of* A, B *defined by (1). Let* $C = [c^-, c^+]$ *be the real interval such that* $C = B \div_{J_c} A$ *is given by (2). Then there exists* $J_{c'} \in \mathcal{F}(\mathbb{R}^2)$ *of* A, C *or* B, C^{-1} *given by (1) with parameters* q' *and* r' *such that the following equivalence holds*

$$C = B \div_{J_c} A \iff \begin{cases} (i) & B = A \odot_{J_{c'}} C, \\ (ii) & A = B \odot_{J_{c'}} C^{-1}, \quad \text{where} \quad C^{-1} = [\frac{1}{c^+}, \frac{1}{c^-}]. \end{cases} \tag{4}$$

Proof The complete proof is extensive, but we focus here on the principal concepts employed to demonstrate this theorem. In the forward direction, we rely on the proof of Theorem 13 and the concept of g-division, which is reversible using standard interval multiplication. Conversely, we apply Proposition 12 along with the notion that one of the multiplications in (4) consistently results in the precise boundaries specified by (3). ■

2.1 Extension to Fuzzy Intervals

In this subsection, we present a remark that links the J_c-arithmetic of real intervals with fuzzy intervals and briefly formalize this extension.

Remark 15 *The invertibility result for intervals does* not *generally extend to fuzzy intervals. Let* $A, B \in \mathbb{R}_{\mathcal{F}}$ *with* α*-cuts* $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$ *and* $[B]_\alpha = q[A]_\alpha + r$ *be fuzzy intervals under the completely correlated joint possibility distribution* J_c *as defined in (1). Applying the invertibility theorem levelwise for each* $\alpha \in [0, 1]$,

$$[C]_\alpha = [B \div_{J_c} A]_\alpha, \quad [B]_\alpha = [A \odot_{J_c} C]_\alpha, \quad [A]_\alpha = [B \odot_{J_c} C^{-1}]_\alpha,$$

may yield intervals $\{[C]_\alpha\}$ *that violate fuzzy interval properties, such as convexity and monotonicity of endpoints.*

3 Conclusion

In this contribution, we present J_c-division and J_c-multiplication for real intervals. These operations are defined by the particular joint possibility distribution J_c called complete correlation. Moreover, we investigate the inverse property between the defined operations. Although the structure of J_c-arithmetic for intervals is elegant and invertible, its naive extension to fuzzy numbers through α-cuts introduces inconsistencies. These findings indicate that the theory of interactive fuzzy arithmetic demands more refined constructions beyond point-wise interval analogies.

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Abstracts

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Uncertainty-Aware Machine Learning: A Case Study on EEG Data

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Reliable machine learning systems must not only make accurate predictions but also express how confident they are in those predictions, particularly in safety-critical domains [3]. Electroencephalography (EEG) data provide a challenging testbed for uncertainty quantification (UQ) due to their high levels of noise and artifacts (e.g., motion or sensor instability), strong inter-subject variability in brain activity patterns, and the tendency of models to produce overconfident outputs under uncertainty [1]. Such data are commonly used in classification tasks that infer cognitive or clinical outcomes from neural signals, including motor imagery, cognitive state, and neurological disorder detection.

This work bridges theoretical uncertainty quantification (UQ) frameworks with their practical deployment in real-world biomedical systems. A case study on uncertainty-aware EEG classification analyzes key paradigms for uncertainty modeling, including Bayesian inference, fuzzy logic, and Dempster–Shafer theory [2], and outlines major challenges and methodological directions toward more trustworthy and reliable decision making using EEG signals.

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Distance functions between closure systems and their connections with fuzzy formal concept analysis

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Closure systems provide a robust mathematical framework for representing and comparing structured knowledge [1]. By measuring similarities or dissimilarities between closure systems, we can assess how closely related two datasets or knowledge structures are [7,6,5,10]. In this study, we introduce and analyze several distance-based measures for quantifying dissimilarities between closure systems, with a particular focus on their application to concept lattices generated over a common set of objects but distinct attribute sets. The proposed approach is based on principles from fuzzy formal concept analysis, extending traditional concept lattice comparison by allowing for graded relationships between objects and attributes [9,3,11,4,8]. This connection enables a richer interpretation of conceptual proximity and partial membership, enhancing the expressiveness of distance functions in real-world, noisy datasets [2].

The proposed algorithms were implemented in the Go programming language and applied to an extensive empirical study of industrial map datasets extracted from world school atlases. Experimental results demonstrate that the introduced distance measures effectively capture structural variations among these concept lattices and reflect their underlying conceptual complexity. The integration of fuzzy logic principles provides an additional interpretative dimension for assessing similarity across school atlas representations.

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d-Choquet Inspired Functions and Their Extension for Adaptive Image Aggregation and Pooling

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Dimensionality reduction is a fundamental task in image processing, especially in scenarios where storage and computational efficiency are critical, such as real-time vision systems or embedded devices. Typical examples include the pooling layers of convolutional neural networks (CNNs), which reduce spatial dimensions while preserving essential information.

In this contribution, we introduce the family of d-Choquet Inspired functions, a novel class of non-additive aggregation operators defined through a dissimilarity function d and an auxiliary function F acting on ordered subsets of the input [1]. These operators generalize the classical Choquet integral by incorporating pairwise dissimilarity information into the aggregation process [4].

We also propose an extension of this model in which F is replaced by a combination of multiple functions F_i, enabling locally adaptive aggregation behaviors that capture heterogeneous relationships among input values. This formulation enhances flexibility, allowing the aggregation process to adapt to local characteristics of the data while preserving the theoretical consistency of the d-Choquet framework.

A comparative study is carried out on several aggregation-based operators for dimensionality reduction of images, including noise-corrupted cases (Gaussian, salt-and-pepper, Poisson, and speckle). The evaluated methods include statistical operators (mean, median, minimum, maximum) [2], the traditional Choquet integral, and both the standard and extended d-Choquet Inspired functions. The study follows the methodology of local image reduction based on aggregation operators proposed in [3].

All methods are evaluated within a sliding-window framework, using metrics such as a mean squared error (MSE), custom similarity measure (SM) and the Structural Similarity Index (SSIM) [5]. The analysis highlights conceptual differences between classical and non-additive aggregation approaches, emphasizing the potential of the d-Choquet Inspired family for adaptive image aggregation. The most promising configurations will be implemented as pooling layers within CNN architectures to further study their behavior and contribution to feature extraction.

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On Fuzzy Approach to the Concept of Value of Information in Optimal Control

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The concept of Value of Information (VoI) is based on the information theory and was initially introduced in the late 60s ([6], [5]), and further developed in a number of works (e.g., see in [4]). For optimal control problems considered as maximization problems when the variables in the profit function are random, the VoI is calculated as the difference between two values: the expected optimal profit assuming that realizations of the random variables are known, and the optimal expected profit when decisions are based only on the probability distribution of the random variables. It is a tool for quantifying the economic value of new information and determining whether the potential improvement in decision quality justifies the cost and effort of obtaining the data.

Although VoI has become increasingly popular in recent years (e.g., [3], [2]), the classical approach has its drawbacks. Its applicability can be severely restricted in situations where stochastic modelling of the problem is difficult or infeasible. To overcome this difficulty, we propose an alternative approach based on fuzzy logic. We develop fuzzy modelling techniques, focused on extending the concept of VoI and intended for application in ecological management (e.g., renewable resource control and extraction). This includes analysis of VoI for related optimal control problems, as well as development of decision-making procedures for assessing the need of acquiring new information (including that obtained as a result of expert evaluation). This includes also a comparison of the proposed approach with known examples (e.g., see [1]) of application of the VoI theory in a fuzzy logic-based framework.

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Exploring the upper n-Sugeno integral and its role in scientometrics

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The classical Sugeno integral remains one of the most prominent nonadditive integrals ever. Many researchers have studied and generalized the Sugeno integral in various directions, aiming to preserve its original desirable properties while enriching it with additional ones suitable for diverse applications. A promising approach, proposed by Boczek et al. in [1], extends the original concept through the introduction of the upper and lower n-Sugeno integrals.

The upper n-Sugeno integral of a measurable function $f\colon X \to [0,\infty]$ with respect to a monotone measure μ on X and an admissible fusion map $\circ\colon [0,\infty]^2 \to [0,\infty]$ is defined recursively by

$$\mathbf{Su}_{n+1}(\mu, f) = \sup_{t\in[0,\infty]} \{(t \circ \mathbf{Su}_n(\mu, f)) \wedge h_{\mu,f}(t)\},$$

where $h_{\mu,f}(t) = \mu(\{x \in X : f(x) \geq t\})$. Building on the results of Chitescu [2], we demonstrate that the computation of the upper n-Sugeno integral can be reformulated as the problem of finding the midpoint of a level measure. This perspective allows us to derive several sufficient conditions for its evaluation. Furthermore, it reveals new connections between Sugeno-type integrals and nonlinear equations arising in informetrics.

In our contribution, the upper n-Sugeno integral plays a central role in introducing the so-called Hirsch-Sugeno operator, which is derived from the upper 2-Sugeno integral by replacing $\mathbf{Su}_1(\mu, f)$ with the Hirsch index, that is, the Sugeno integral with respect to a counting measure [3]. We discuss potential applications of the Hirsch-Sugeno operator as a new scientometric tool. We show that this operator unifies a broad spectrum of existing indices and their modifications, offering a flexible and conceptually coherent mathematical framework for assessing scientific impact within the theory of fuzzy integrals.

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Fuzzy relations in some decision-making and expert systems

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Fuzzy relations are carriers of imprecise relationships in models of a decision-making or expert system. They enable the formal representation of human preferences, dependencies, and subjective judgments.

An important application of relations in decision-making processes is through preference relations, which capture the subjective preferences of the decision maker. The decision-making process is then guided by the aggregation of these preference relations. In expert systems, fuzzy relations model expert judgments, or they may replace or supplement rules.

Our aim is to consider relations at other levels of modeling, for instance, at the level of the universe. One such example is to consider a value of a linguistic variable expressing temperature and the relation between temperature degrees, and then, based on that information, construct a modified value of a linguistic variable. The construction of such modifiers (hedges) is provided by De Cock and Kerre in [1].

Moreover, it is possible to study the influence of relationships among experts in multi-round decision or voting systems, for example using the Delphi method – an iterative process of systematically collecting and aggregating expert opinions to achieve consensus under conditions of uncertainty.

We aim to select an appropriate method and investigate the role of relations in systems modeled by fuzzy sets and their generalizations, such as interval-valued or intuitionistic fuzzy sets.

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Uncertainty in Idea Management with Elements of Game Theory

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In [3], idea management is a sub-process of innovation management focused on systematically generating, evaluating and selecting ideas to turn those ideas into actionable solutions. It structures the flow and decision-making around ideas to increase the efficiency and success of innovation implementation. One of the stages of the idea management process is the acceptance, deferment and rejection of ideas. Idea generators (for example, employees of a company) are viewed here as a single player who generally offers many ideas. Idea acceptors (for example, company managers) are the second player who makes a decision on each proposed idea. They may have common goals, but they may also have different opinions. The payoffs of these players are generally uncertain. We will develop a possible interpretation and solution for decision making from a game theory perspective.

Game theory is a crucial tool for modeling competitive interactions among decision-makers in various fields, including engineering, economics and idea management. Classical game theory assumes complete information, where all payoffs are known with certainty. Evaluating ideas is associated with uncertainty, making classical methods inadequate. We propose to evaluate ideas using extensive-form games with uncertain payoffs (see [4]) or bi-matrix-form games with uncertain payoffs. One option is to describe the uncertain payoffs with uncertain intervals. Various descriptions of such an approach can be found in the literature, see [5,1,2] and the references cited therein. The paper [2] concluded that the idea generators themselves play a significant role in the evaluation of ideas. The main issue of the article presented here is finding Nash equilibrium under uncertainty and interpreting the resulting equilibrium in the context of idea management.

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The maximal set of non-isomorphic lattices represented by Δ-incidence matrices preserving $\{0,1\}$-aggregation

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It is perfectly natural that nowadays we want to perform all computations using computers. However, especially in mathematics, we may encounter structures that are difficult to describe in a way that a computer can understand. One such structure is a lattice, which is an ordered set in which every pair of elements has a well-defined minimum and maximum. The aim of this paper is therefore to represent lattices using matrices, which computers are very good at working with, and then perform aggregations with them. For example, given two binary matrices of the same dimensions, we define their minimum and maximum as new matrices whose entries are the element-wise minimum or maximum – corresponding to logical conjunction (AND) and disjunction (OR), respectively. We apply such aggregation operations – particularly MIN and MAX – to matrices representing lattices of increasing size, starting with two-element lattices and progressing to larger ones. We will demonstrate that this task is not as straightforward as it might initially seem, as a number of challenges arise that even modern computational tools struggle to overcome. On the other hand, this approach opens up a completely new possibility: until now, we have only been able to perform aggregation within a single, specific lattice. But now, we can aggregate across different lattices themselves. We prove that for both the minimum and the maximum, there exist maximal closed sets of lattices, from which it follows that the 'lattice of all equally-sized finite lattices' does not exist.

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Differential Equations with Respect to Fuzzy Measures: Basic Concept and Modeling

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Differential equations with respect to fuzzy measures could be described as a generalization of ordinary differential equations. The first step to the differential equations with respect to general fuzzy measures was the case of the distorted Lebesgue measures, from [5]

$$\frac{\partial y(\tau)}{\partial \mu_m} = F(\tau, y(\tau)),\ y(0) = 0.$$

Those equations were studied in [3,5], where basic properties were proven. The generalization of those equations (the distorted Lebesgue measure was changed to a general fuzzy measure) was provided in [2]

$$\frac{\partial y(\tau)}{\partial \mu} = F(\tau, y(\tau)),\ y(0) = 0.$$

In the first part of our post, we will introduce the basic concepts necessary to understand the differential equations with respect to fuzzy measures. We will start with the concept of fuzzy measures [6] and with the Choquet integral [1], and we define so-called Choquet-Radon-Nikodym derivatives [4] as a generalization of well-known Radon-Nikodym derivatives.

In the end, we will use this concept for modeling a dynamical system on real data. On this model, we will show that ordinary differential equations are not enough for some dynamical systems, and so we will show the necessity of this concept.

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A note on some inequalities for Birkhoff weak integrable functions

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The Minkowski and the Hölder inequalities play an important role in many areas of pure and applied mathematics, such as Convex Analysis, Probabilities, Control Theory, Fixed Point theorems, Mathematical Economics [13,12,11,9,6]. Also, non-additive measures, non-additive integrals and set-valued integrals are useful tools in several areas of theoretical and applied mathematics [CGIS,C8,7,2,3,5,1,4].

In this talk we present and we prove some Hölder and Minkowski inequality (or reverse inequality) types obtained for Birkhoff weak integrable functions with respect to a non-additive measure. At the end some example and applications are given.

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Continuity and Irresoluteness through operations between fuzzy bitopological spaces

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The recent articulation of the $(i,j)^*_\gamma$ -operation within the context of a fuzzy bitopological space [1] represents the latest advancement in research pertaining to the operational approach. In this treatise, our principal objective is to scrutinize the characterization of functions within a designated fuzzy bitopological space via an operational methodology. To this end, we initially profound the concept of the $(i,j)^*_\gamma$ -fuzzy open set. A selection of properties pertinent to this newly introduced set is examined by inaugurating concepts of neighborhood and quasi-coincidence. Few properties of newly introduced set are studied by initiating neighborhood and quasi-coincidence. Application of $(i,j)^*_\gamma$ -fuzzy open sets are shown by producing results on continuity and irresoluteness through operation approach.

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Capturing Temporal Interrelationships Among Data by Choquet-like integrals: Application to Network Traffic Forecasting

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Next-generation networks, such as 6G and IoT, require adaptive traffic prediction methods that respond effectively to dynamic environments than traditional models, which rely on extensive historical data dependency [6]. This work introduces the concepts of temporal capacities and temporal Choquet-based integrals and proposes a methodology based on temporal Choquet integrals and C_F-integrals [1,4], prioritizing recent data and capturing temporal dependencies within sliding windows. We show that temporal Choquet integrals are lower 0-bounded aggregation functions in $[0,+\infty[$, and, under certain constraints, temporal C_F-integrals are lower 0-bounded pre-aggregation function in $[0,+\infty[$ [3]. We also proved some important properties.

The proosed approach improves prediction flexibility and accuracy by integrating temporal capacities and power-based variations. Experiments comparing 17 formulations of temporal Choquet and C_F-integrals show superior performance over the Poisson Moving Average [2], especially with optimized window sizes. These results demonstrate the potential of fuzzy temporal aggregation for proactive network traffic prediction.

The definitions of the different functions F used in the experiments are available at `https://github.com/dennerguti/temporalChoquet`. In particular, the overlap function $\mathrm{FBPC}(x, y) = xy^2$ [5], for any $x, y \in [0, 1]$, allowed for an excellent behavior of the respective temporal C_{FBPC} when applied to predict network traffic.

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Modifying Decomposition Integrals Using Inequalities

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Decomposition integrals are a powerful framework of non-linear integrals that include many well used integrals in theory and practice, such as the Choquet integral, the Shilkret integral, the PAN integral, and the concave integral.

A collection is a non-empty subset of the power set of the non-empty finite space X excluding the empty set. A decomposition system, usually denoted by $\mathcal{H}$, is a non-empty set of collections. A decomposition integral with respect to the decomposition system $\mathcal{H}$ is an operator $\mathsf{dec}^{\mu}_{\mathcal{H}} \colon \mathcal{F} \to [0, \infty[$, where $\mathcal{F}$ is the space of non-negative functions defined on X and μ is a monotone measure on the power set of X, defined by

$$\mathsf{dec}^{\mu}_{\mathcal{H}}(f) = \bigvee_{\mathcal{D} \in \mathcal{H}} \bigvee \left\{ \begin{array}{l} \sum\limits_{A \in \mathcal{D}} \alpha_A \mu(A) \, : \\ \sum\limits_{A \in \mathcal{D}} \alpha_A \chi_A \leq f, \\ \alpha_A \geq 0 \text{ for all } A \in \mathcal{D} \end{array} \right\}$$

In the presented work, we expand this framework by introducing a possibility to include new linear inequalities to the definition of decomposition integrals.

These integrals possess standard properties of decomposition integrals, i.e., non-decreasingness and continuity. Positive homogeneity, a property of decomposition integrals, is not guaranteed for this modification, in general. A usability of this new class of integrals is illustrated by some figurative examples.

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Bridging Extensionality and Similarity-Based Reasoning

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Similarity-based reasoning systems place the notion of similarity at the center of their motivation and design. Yet similarity is not only a general conceptual idea but also a specific family of binary fuzzy relations that act as fuzzy equalities. And their natural link to the concept of extensionality in fuzzy settings motivates to bridge extensionality and similarity-based reasoning. Following the work [2], this contribution explores how the extensionality shapes the behavior of similarity-based reasoning fuzzy inference systems. For this purpose, the *extensional hull* [1] is considered and a dual notion of an *extensional base* [2] is introduced. The first property examined within this framework is robustness, which proves essential for understanding how these systems respond under small perturbations. At the same time, several open questions emerge. This contribution shows the potential of the foreshadowed approach and communicates the open questions for a wider discussion with the goal towards the development of a reliable and interpretable intelligent inference schemes.

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Some ideas about extending IOWA operators to multivariate domain

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The aggregation of information frequently entails sorting data according to multiple criteria, particularly when it is advantageous to arrange the data based on an auxiliary attribute. In such cases, induced aggregation functions, such as IOWA operators [4], are commonly employed. Data is not always formed by singletons; rather, information under consideration comprises several attributes, variables, dimensions or coordinates. In order to adress the sorting of multivalued structures, several techniques have been proposed in the literature: admissible orders [4] (total orders that extend a partial order), admissible permutations [3] (in order to represent the set of possible arrangements obtained from admissible orders) or degree of totalness [3] (to identify a set of the best admissible permutations that are maximal with respect to a given partial order). So far no induced aggregation operators have been studied in the multivalued context. We introduce multivalued induced aggregation functions as a extension of classical induced aggregation operators. In these operators, both the data to be aggregated and the inducing criteria are represented by multivalued entities, which may, but need not, share the same underlying structure or dimensionality as the data being aggregated. We propose a new general definition of IOWA operators and three new methods for obtaining these operators. We present new results and ways of constructing these operators, and we study their relevant properties and the relationships among them.

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Relationships between bounded lattice structures and uninorms construction methods.

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The study of aggregation functions within fuzzy logic has given rise to several families of operators, including triangular norms, triangular conorms, uninorms, and nullnorms [3,2,1]. Initially, these operators were defined on the unit interval $[0,1]$, but subsequent research extended them to more general frameworks such as arbitrary closed intervals $[a,b]$, lattices, product lattices and other ordered structures [7,6]. A recurring topic in this area is the problem of understanding how aggregation functions can be consistently defined in these broader domains while preserving the essential algebraic and logical properties that motivate their use.

Here, we focus on the study of uninorms over bounded lattices and on the relationship between the structure of the bounded lattice and some examples of aggregation functions [10,9,8]. In the literature, we can find several papers in which authors have focused on defining construction methods of very specific uninorms on very general bounded lattices [9,4]. In some of these cases, the methods are quite universal and applicable to any bounded lattice, but they often result in limited families of uninorms.

Our research direction is the opposite: by restricting our attention to particular lattices we aim at defining more general families of uninorms. More precisely, we focus on the subset of incomparable elements with the neutral element [5] and, by considering on that subset an auxiliary triangular norm, triangular conorm or ever uninorm, we present a new construction method for uninorms on bounded lattices.

This approach not only generalizes some previous construction methods, but also offers greater flexibility and structural insight. It reveals how the local behavior around the neutral element can determine global properties of the aggregation function. We illustrate the method with examples on specific bounded lattices and discuss how different choices in the auxiliary aggregation function impact the resulting uninorm. This framework opens new directions for further exploration of aggregation operators adapted to complex algebraic structures.

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Fuzzy Measures and Choquet Integral: A Framework for Uncertainty Quantification in Machine Learning

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Uncertainty quantification is a central challenge in machine learning, essential for developing reliable and trustworthy predictive models [2]. Typically, uncertainty is decomposed into Aleatoric Uncertainty (AU), arising from inherent randomness in the data generation process, and Epistemic Uncertainty (EU), reflecting the lack of knowledge due to limited data or model capacity [3]. Their combination defines the Total Uncertainty (TU).

Recently, a family of statistical divergences known as Integral Probability Metrics (IPMs) has been extended to compare Imprecise Probability (IP) models [1], leading to the notion of Imprecise Integral Probability Metrics (IIPMs) [4]. Formally, for a class of continuous real-valued measurable functions $\mathcal{F}$ bounded by a positive constant $c > 0$, i.e., $|f| \leq c$ for all $f \in \mathcal{F}$, and for two fuzzy measures ν_1, ν_2, an IIPM is defined as

$$\sup_{f \in \mathcal{F}} \left| \oint f \, \mathrm{d}\nu \; - \; \oint f \, \mathrm{d}\nu_2 \right|, \qquad (1)$$

where $\oint$ denotes the (asymmetric) Choquet integral. In particular, this expression provides a natural way to quantify EU as the distance between a lower probability and its conjugate upper probability [4].

In this work, we build upon these developments and derive explicit formulations of AU, EU, and TU using IIPMs (Eq. (1)), offering a unified approach to representing and quantifying uncertainty through IP models and Choquet Integral. We focus specifically on the case of supermodular fuzzy measures, as most IP models fall within this category. However, the expressions we present are valid and justified for the general class of fuzzy measures. Finally, we discuss the computational aspects of the proposed methodology.

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Asymptotic methods in statistic with imprecise data

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Analyzing imprecise experimental data requires mathematical tools capable of handling two types of uncertainty: randomness and imprecision. Classical statistical methods effectively address randomness, but imprecision in data is usually represented using intervals or fuzzy sets. The main challenge in developing statistical tests or confidence intervals for such data arises from the fact that the traditional concept of a probability distribution (e.g., normality) does not naturally extend to random intervals or random fuzzy numbers. Consequently, most existing approaches have relied on bootstrap methods [3] or permutation techniques [1].

In this presentation, we demonstrate how the limiting distribution of random elements associated with imprecise data can be derived under standard assumptions and mild regularity conditions. Simulation studies across a range of data configurations confirm the asymptotic results and reveal strong, even small-sample performance (see, e.g., [2]).

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Weighted Evidential Divergence Measure based Method for Multi-View Information Fusion

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Multi-view feature learning-based approaches have advanced significantly in a wide range of applications [7, 6]. These methods are developed over complex models integrating diverse view features obtained through deep neural networks (DNNs). Traditional multi-view learning-based approaches often provide equal weights for multi-views or learn the weighting factor for each view [6, 5]. Although existing methods have attempted to improve classification performance, they often ignore uncertainty in decision-making problems such as the classification of noisy or corrupted images with multiple views when employed to diagnose medical disorders [4] or RGB-D scene recognition [5]. For instance, in multi-modality-based DNN methods for the RGB-D scene recognition, the algorithm may require outputting a confident recognition based on an RGB image for one scene. In contrast, for another scene, a depth image may be sufficient. Therefore, adapting dynamic correlation factors while modeling multi-view classification is essential. This study focuses on developing a weighted evidential divergence measure-based approach by estimating uncertainty during the multi-view classification. This facilitates an intuitive approach for fusing the multi-view information by incorporating the diverse views at the evidence level. Initially, we compute the Dirichlet distribution for each view inspired by the method presented in [3], followed by estimating the belief and uncertainty masses. After estimating belief and uncertainty mass for each view, a systematic fusion approach based on a weighted evidential divergence measure is designed to compute the overall belief and uncertainty masses. Subsequently, Dempster's combination rule is incorporated to calculate the final belief and uncertainty mass for decision-making. The comprehensive experimental analysis of the proposed approach is conducted on two publicly available SUN RGB-D and NYU Depth V2 datasets [2, 1] for scene recognition, compared with the existing techniques, to validate the efficacy of the proposed approach.

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An Overlap Function-Induced Interactive Feature-Weighting Improved Oversampling Method for Imbalanced and High-dimensional Data Classification

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Over the past few decades, research on high-dimensional and imbalanced data classification using machine learning has gained considerable attention among researchers [1-4]. A common approach to deal with these classification approaches is to employ feature selection for dimensionality reduction and oversampling to balance class distributions [1-4]. Existing feature selection strategies are based on the determination of relevant features using the Gini index [1-4], which evaluates the class separability of the features using the global distribution across all samples. This measure overlooks individual sample overlap between different classes with respect to specific feature and fails to capture feature interactions. The traditional oversampling method, specifically Synthetic Minority Over-sampling Technique (commonly abbreviated as SMOTE) determines the neighbourhood of minority samples using the Euclidean Distance (ED) [5,6] metric. However, this metric implicitly treats all features as equally important, which becomes less effective in high-dimensional settings [6]. In response, features are frequently weighted using metrics, such as the Fisher score or correlation score, when assessing distances between minority neighbourhoods; but, this approach also fails to capture interaction information among features [6]. To overcome the aforementioned issues, we propose a novel overlap-guided interactive feature weighted oversampling method. First, a feature selection strategy is developed by integrating the bivariate feature interaction with sample-level overlap information. For this purpose, we introduce an overlap function [7] based interactive measure for every feature subsets. Subsequently, we compute individual feature scores. Next an overlap-induced interactive measure-based improved SMOTE operator is introduced to define the neighbourhood of minority class samples. This approach offers a precise definition of neighbourhood by emphasizing feature importance while considering feature interactions and sample-level overlap information. The efficacy of the proposed technique is assessed utilizing publicly available high-dimensional and imbalanced data set [4-6] and compared with existing methods reported in the literature.

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Modeling and Application of Quadri-Polar Fuzzy Fantastic Ideals in BCI-Algebras Using the TOPSIS Method

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A quadri-polar fuzzy set ($q\mathcal{P}$-$\mathcal{F}$ set) generalizes the conventional fuzzy set by introducing four distinct membership degrees, allowing a richer representation of uncertainty and vagueness. In this study, we introduce the concept of quadri-polar-(ϖ, ϑ)-fuzzy fantastic ideals ($q\mathcal{P}$-(ϖ, ϑ)-$\mathcal{FFI}(s)$) in BCI-algebras, constructed on the foundation of the $q\mathcal{P}$-$\mathcal{F}$ framework. We further define quadri-polar-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-fuzzy fantastic ideals ($q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}(s)$) and establish necessary and sufficient conditions for an $\in_{\tilde{\sigma}}$-$q\mathcal{P}$-$\mathcal{F}$ set and a $q_{\tilde{\tau}}$-$q\mathcal{P}$-$\mathcal{F}$ set to qualify as quadri-polar fuzzy ideals ($q\mathcal{P}$-$\mathcal{FI}$) in BCI-algebras. In addition, a $q\mathcal{P}$-$\mathcal{F}$ TOPSIS framework is developed for multi-criteria group decision-making (MCGDM) as a natural extension of the classical TOPSIS approach, enabling the ranking and selection of optimal alternatives under quadri-polar fuzzy positive and negative ideal conditions. Finally, illustrative examples

are provided to demonstrate the effectiveness and real-world applicability of the proposed $q\mathcal{P}$-$\mathcal{F}$-TOPSIS method.

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On Archimedean t-Norms Based on Piecewise Linear Additive Generators

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Triangular norms (t-norms) represent a fundamental class of binary operations used to model conjunction in fuzzy logic and fuzzy set theory. They are defined as monotonic, commutative, and associative functions with 1 as the neutral element on the interval $[0, 1]$. T-norms generalize the classical logical AND to continuous truth values and provide a formal framework for reasoning with imprecise information, cf. [3].

Among the various types of t-norms, continuous Archimedean t-norms play a particularly important role due to their continuity and characteristic Archimedean property, no value in the open interval $(0, 1)$ remains invariant under repeated application. This property ensures a smooth weakening of truth values and natural behaviour of conjunctions in uncertain framework.

A key approach to studying these t-norms is their representation via additive generators, which allows a unified and analytically tractable description of the entire class of Archimedean t-norms. An additive generator $f\colon [0, 1] \to [0, \infty]$ is a continuous, strictly decreasing function satisfying $f(1) = 0$, for which the t-norm T can be expressed as

$$T(x, y) = g\Big(f(x) + f(y)\Big),$$

where g is the pseudo-inverse of f. This concept, introduced in the seminal works of Schweizer and Sklar [1], of Ling [2], and monograph [3], has become a cornerstone of both the theoretical classification of t-norms and the construction of new operators with desired properties.

The aim of this contribution is to highlight the relationship between the piecewise linearity of the additive generator and the associated Archimedean t-norm.

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Representing Ordered Functional Weighted Averaging Operators

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Ordered functional weighted averaging operators (OFWA operators for short) were introduced in [2] and they represent a generalization of OWA operators commonly used in decision making. The original definition of an n-ary OFWA operator uses n different functional weights satisfying the condition that their sum is equal to 1. Due to this condition, it is easy to see that every OFWA operator belongs to the class of intermediate functions, i.e., functions that are between the min and max functions. We show that the converse is also true, i.e., that every intermediate function can be represented as an OFWA operator, cf [1]. Moreover, the mentioned representation can be found using two non-zero functional weights, i.e., using only one independent function from $[0,1]^n$ to $[0,1]$. Consequently, the former definition of OFWA operator can be significantly simplified. This naturally determines a correspondence between the class of all n-ary functions acting on the unit interval on the one hand and the class of all n-ary intermediate functions (OFWA operators) on the other.

We describe some properties of this correspondence. From the topological point of view, the family of all functions from $[0,1]^n$ to $[0,1]$ forms a metric space with respect to sup-norm and the class of all intermediate functions is a subspace of this metric space. We show that the considered mapping is uniformly continuous and it maps continuous functions to continuous OFWA operators. As an algebraic structure, the family of all functions acting on $[0,1]^n$ forms a complete lattice with respect to the pointwise operations of supremum and infimum and the same holds for the class of intermediate functions. The considered correspondence is a complete lattice homomorphism; moreover, it maps aggregation functions to the set of idempotent aggregation functions.

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Fuzzy Extensions of Antiideal Structures in Semigroups

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This presentation investigates the fuzzification of antiideal-based structures within the framework of semigroup theory, with the primary goal of extending classical algebraic concepts into the fuzzy domain. In traditional semigroup theory, antiideals and bi-antiideals are special subsets that deliberately avoid closure under the semigroup operation, thereby standing in contrast to the well-established notion of ideals, which are closed under such operations. These antiideals serve as a useful counterpoint to ideals and help in analyzing the structural behavior of semigroups from a different perspective.

Building upon this foundational idea, we introduce a new class of subsets termed (m, n)-bi-antiideals, which generalize the concept of bi-antiideals by incorporating parameters m and n that govern the structure of the elements involved. We explore the defining properties of these (m, n)-bi-antiideals, investigate their algebraic behavior, and analyze how they relate to previously studied constructs within the theory.

The study then transitions into the fuzzy domain, where uncertainty and partial membership play a central role. We define fuzzy antiideals and fuzzy (m, n)-bi-antiideals using membership functions and further analyze these fuzzy structures through the lens of level sets. These fuzzy extensions allow for the modeling of algebraic systems that are not strictly binary in nature, thus offering a richer and more nuanced framework suited for real-world applications where imprecision is inherent.

Furthermore, we establish meaningful relationships between the classical (crisp) and fuzzy variants, demonstrating how fuzzification not only preserves essential algebraic properties but also adds layers of interpretability and flexibility. Overall, the results contribute to a more comprehensive understanding of semigroups under uncertainty and lay the groundwork for future research in fuzzy algebraic systems and their applications in computational and theoretical contexts.

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Independence of fuzzy events

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The independence of events A, B means that the conditional probability is the same as the unconditional one, $P(A|B) = P(A)$, or, equivalently, $P(A \wedge B) = P(A) \cdot P(B)$. For fuzzy events, $A \wedge B$ is a fuzzy intersection based on the pointwise application of a t-norm. The choice of the t-norm gives rise to many notions of independence and conditional probability [7].

In classical probability, events A, B are independent if and only if A, B' are independent, etc. For fuzzy sets, this equivalence does not necessary hold. Therefore, we introduce **strong independence**, requiring that all pairs (A, B), (A, B'), (A', B), (A', B') are independent. For observables, i.e., generalized random variables, the notions of independence and strong independence coincide, though they are still dependent on the choice of the fuzzy intersection.

The next question is which fuzzy event structures possess non-trivial examples of (strongly) independent observables. This holds for the product intersection. For other fuzzy intersections, strong independence appears to be a very restrictive assumption [6,5]. It thus enables proofs of analogues of classical theorems, which apply only to special cases.

Among MV-algebras, only those equipped with product (see [2]) admit the strong independence [7].

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Independence of fuzzy observables

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In the probability theory of many-valued algebras, events are represented by fuzzy sets or by elements of an algebra (MV-algebra, effect algebra, etc.), $\mathcal{A}$. The notion of a random variable is generalized to an *observable*, which is a σ-homomorphism of the Borel σ-algebra into $\mathcal{A}$.

Statistical results like the central limit theorem, laws of large numbers, etc., are formulated for sequences of independent observables. The independence of observables x, y is defined by the condition $P(x(A) \wedge y(A)) = P(x(A)) \cdot P(y(A))$ for each Borel set A, where $\wedge$ represents a conjunction (resp. a fuzzy intersection) in the algebra (resp. in the collection of fuzzy sets) $\mathcal{A}$. Thus, there are many possible notions of independence. Several of them have been treated in previous works and—surprisingly—generalizations of classical theorems were successfully derived in all of them [3]. We tried to explain this strange phenomenon and uncover the principles behind the proofs. We have found out that some assumptions of independence are so strong that they deny almost all fuzziness. E.g., in terms of fuzzy sets,

$$\{\omega : 0 < x(A)(\omega) < 1, 0 < y(A)(\omega) < 1\}$$

is of zero probability. To avoid this degenerated case, we must take the product for $\wedge$. In terms of an MV-algebra, we need an *MV-algebra with product*, studied in [3]. The presence of a product—and its role in defining conditional probability and independence—appears crucial for a successful development of probability and statistics on fuzzy events.

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Linguistic summaries in weights adjustments and criteria reduction for evaluation and multicriteria decision making

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In evaluation of entities and in multicriteria decision making, the domain expert might face a higher number of criteria. Usually, the criteria are not fully independent. Hence, the initial weights assigned by the domain expert in multicriteria decision making should be adjusted. In evaluation (or database queries), significantly dependent criteria might be excluded. CRITIC is a method that adjusts the weights considering the correlation between the criteria [1]. However, this is a symmetric adjustment. We propose classical prototypes of linguistic summaries with restrictions, that can recognize the direction of dependency [2, 3]. We evaluate the validities of summaries most of small changes in criteria A causes small change in criteria B and vice versa for all pairs of criteria. From the matrix of dependencies between criteria, we compute by aggregation functions which criteria are significantly influenced by others and which significantly influence others. Consequently, we adjust weights or exclude significantly dependent criteria. This approach is applicable for numerical data, categorical data, and vague data represented by fuzzy numbers. Also, small imprecision in data is attenuated. However, in conjunctive evaluation, the weights are not normalized, but handled by implication [4, 5]. Adapting weights for this task is a topic for further research. The results are demonstrated in an illustrative example. The application fields range from evaluation of companies with respect to economic and ecological criteria, to examining citizens affected by poverty and proposing the most suitable product or service.

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Aggregation operators and algebras for musical data fusion

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We present a class of aggregation operators developed to model complex aspects of human sensual perception, particularly in music. Unlike standard global aggregation functions, cf. [4], these operators do not necessarily adhere to traditional boundary conditions. Instead, their definition is motivated by experimental results in sound and light perception and key psychoacoustic laws. For instance, the perception of pitch diapason or consonance in an orchestra depends on the number of playing instruments (loudness) and is highly individual and fuzzy. This perception adheres to the Weber-Fechner law, necessitating logarithmic scaling for perceptual axes. Crucially, a suitable aggregation operator must satisfy associativity and commutativity, meaning the resulting sound should not depend on the order or manner in which the inputs (e.g., violins) are grouped. Standard aggregation methods, such as the arithmetic mean of slight mistuning values, fail this requirement, producing different results depending on the grouping.

Furthermore, the contribution explores the psycho-acoustical properties of tone systems, cf. [2], as a strong motivation for studying the algebraic structure of commutative and associative aggregation operators. Traditional attempts to define operations over tone systems based solely on either psychological (qualitative) or physical (quantitative) attributes have proven insufficient. We propose an integrative approach to construct pitch functions as composite vector functions. This methodology inherently unifies the qualitative (perceptual) and quantitative (physical) dimensions of sound. Crucially, this integration necessitates the introduction of a fuzzy set structure to refine the mathematical definition of the tone system, cf. [3] and [1]. By leveraging the principles of commutative and associative aggregation operators, we create robust algebraic frameworks that accurately model the fusion of acoustic inputs into the listener's coherent musical perception.

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Incorporating temporal aspects into the Choquet–Stieltjes integral

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Our contribution builds upon the foundational theory of the Choquet-Stieltjes (ChS) integral, originally introduced in [5]. This integral provides a measure of expected value with respect to a non-additive measure. Recall that the standard Choquet-Stieltjes integral is defined as:

$$\mathbf{ChS}(f,\mu,\varphi) = \mathbf{Ch}(f,m_\varphi) = \int_{[0,\infty)} \mu(\{f \geq t\})\,\mathrm{d}\varphi(t), \quad (1)$$

where $\mathbf{Ch}(f,m_\varphi)$ denotes the standard Choquet integral, cf. [4] and f is a Σ-measurable nonnegative bounded function. Here, μ is a monotone measure, and m_φ is the Lebesgue-Stieltjes measure associated with a nondecreasing function $\varphi\colon (-\infty,\infty) \to [0,\infty)$, cf. [3].

We introduce dynamic and contextual aspects into this framework by employing the concept of a parametrized level measure [2] and its strict version [1]. These measures are defined, respectively, by the following expressions for $t \in [0,\infty)$:

$$\boldsymbol{\mu}_{\mathcal{A}_{\bullet|\bullet}}(f,t) = \sup\Big\{\mu_t(E)\colon \mathsf{A}_t(f|E) \geq t,\ E \in \mathcal{E}_t\Big\}, \quad (2)$$

$$\boldsymbol{\mu}^S_{\widehat{\mathcal{A}_{\bullet|\bullet}}}(f,t) = \inf\Big\{\mu_t(E^c)\colon \widehat{\mathsf{A}}_t(f|E) \leq t,\ E \in \mathcal{E}_t\Big\}. \quad (3)$$

Crucially, the functions defined in (2) and (3) incorporate the time variable t into all three components of the underlying structure: the measure (μ_t), the aggregation operator (A_t), and the collection of measurable sets ($\mathcal{E}_t$). Applying the standard integration procedure for the Choquet–Stieltjes integral to these parametric level measures leads to the formal introduction of the upper and lower $\mathcal{A}$-Choquet–Stieltjes integrals. These resulting functionals possess interesting theoretical properties and establish a suitable structure for potential future applications, particularly in dynamic decision modeling.

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Missing data imputations through a new family of aggregation functions

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Multiple data fusion techniques can be found in the literature. One well-known family of fusion functions is that of weighted aggregators, which include the ordered weighted averaging (OWA) [3] and its extensions, such as Induced OWA [4]. Both OWA and IOWA have been used to great success in the past [2]. However, weighted averaging operators rely on the distribution of the datasets [1]. Normality is a property heavily desired by these operators, yet it is not commonly held when using real data. Lacking this property usually leads to underperforming results. We believe that the root cause of this problem is due to the presence of duplicated data inputs and that, by eliminating them, a new family of operators can be introduced, improving the effectiveness when working with datasets that deviate from normal distributions. To this end, in this study, we extend Induced OWA operators. The new extension's objective is to better adapt to non-linearly distributed data inputs. We present its definition, along with some of its properties. Lastly, a missing data imputation framework has been used to test the performance of this new family of operators and compare it against that of OWA and IOWA operators.

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Fuzzy Equivalence Relations versus Fuzzy Metrics

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We examine the relationship between fuzzy metrics [2] and fuzzy equivalence relations [3], focusing on the topological properties. We study topologies (introduced in [1]) generated by fuzzy equivalence relations, which are reflexive, symmetric, and T-transitive fuzzy relations, where transitivity is defined with respect to a specific t-norm T, as well as topologies induced by fuzzy metrics. A fuzzy metric is a mapping $M : X \times X \times [0,\infty) \to [0,1]$ that is reflexive, symmetric in (x,y), left-continuous in its third argument t, and satisfies the transitivity condition $T(M(x,y,t), M(y,z,s)) \leq M(x,z,t+s)$ for all $x,y,z \in X$ and $t,s \in [0,\infty)$. In both cases, the topologies are constructed from neighbourhood bases at points, similarly to the approach used for metric spaces. In the work we prove that, the topology generated by a fuzzy metric coincides with the topology induced by a T-equivalence. To do that we define the fuzzy metric as $M(x,y,t) \equiv E(x,y)$, we can simply ignore the argument t and induce the same topology as for E. Furthermore, if M is a strong fuzzy metric, then the induced T-equivalence $E(x,y) = M(x,y,t_0)$ for any fixed $t_0 > 0$ defines the same topology as that generated by M. More generally, we can define T-equivalence as $E(x,y) = \sup_{t>0} M(x,y,t)$ to get the desired outcome. This indicates that, from a topological point of view, we can work directly with fuzzy equivalences, and introducing fuzzy metrics as a further generalization does not lead to fundamentally new topological structures. The important thing is that these topologies are dependant on the t-norms which are used for defining T-transitivity.

The work also introduces the concept of T-dense complete join-semilattice ordered monoids and establishes how these structures provide a natural setting for defining topologies generated by monoid-valued equivalences. Further we study this correspondence into a lattice-theoretic setting, showing how lattice operators preserve and reflect key topological properties of T-equivalences. It is shown that under infinite $\vee$-distributivity of the t-norm, the family of fuzzy balls $\{B_E(x,\alpha)\}_{x\in X}$, where $B_E(x,\alpha) = \{y \in X | E(x,y) > \alpha\}$, forms a neighborhood system.

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Relational modifiers and Delphi method

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Given a fuzzy set that represents a value of a linguistic variable (temperature, age, speed, etc.) by a modifier we understand a mapping that in some way transforms its shape. We deal with relational modifiers, which take into account a given binary fuzzy relation R on the universe X.

Two such classes have been introduced and studied in [5] and [3]. Here the relational modifier of the mapping f is given either by the formula

$$M_R^C(f)(x) = \sup\{C(R(x,y), f(y)), y \in X\},$$

where C is a conjunctor on the unit interval, or by

$$M_R^I(f)(x) = \inf\{I(R(x,y), f(y)), y \in X\},$$

where I is an implicator on the unit interval.

However, this attitude does not completely correspond to real life situations, like joint evaluation by a group of experts with known mutual relations. Namely, if we assume the reflexivity of R (which is indeed a natural assumption - an evaluator should be personally consistent), for conjunctors we come to an expansive modifier, i.e. $M_R^C(f)(x) \geq f(x)$ for all $x \in X$ and for implicators to a restrictive modifier, fulfilling the opposite inequality.

Instead, we make use of the Delphi method (see [4,1]), also known as the ETE (Estimate - Talk - Estimate) technique. Here, the expert decision can be represented, after a suitable normalization, as a mapping $f : X \to [0,1]$, where X, the universe, is the set of all experts. Hence, an evaluation can be identified with a fuzzy set on the universe X. The change of the evaluation may be then understood as a result of a modification, as described in [2]. We can assume that the possible change of the evaluation in the second round of Delphi technique may be caused not only by the strength of the arguments while explaining the individual evaluations, but also by the strength of mutual acceptance among the experts. This can be represented by a binary fuzzy relation on the set X. In the first round a panel of experts evaluates a product individually by all experts. Next, they reveal and explain their evaluations. Finally, the evaluation is repeated, when all the experts may, but need not alter their original evaluations.

The aim of our research is to provide a class of relational modifiers that are not necessarily expansive or restrictive using suitable aggregation functions, namely OWA operators, and study their properties, namely in the process of repeated evaluations.

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A Fuzzy Relation Aggregation Approach to Weighting Experts‘ Opinions

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When combining various sources of information the challenge often lies in how to assign appropriate weights to each source. For example, in a boardroom investment decision, it might seem intuitive to treat every member's opinion as equally important. However, when additional stakeholders such as department directors are involved, their input may justifiably carry different weights.

Simply ranking experts by relative importance is insufficient; it is essential to assign precise numerical weights to their judgments. But how can these weights be determined systematically? Traditionally, this process is metaphorically described as "looking at the current constellation positions," implying a somewhat intuitive or heuristic approach.

In this work, we propose a rigorous method for calculating weights in the context of stock market decision-making using fuzzy aggregated equivalence relations. Specifically, we focus on determining the weights for three experts' assessments of preferred stock operations: namely, buy, hold, or sell decisions.

Our objective is to identify weight values such that, when applied within the aggregated equivalence relation function, the resulting consensus measure closely aligns with outcomes observed retrospectively. This approach enables a more objective and mathematically grounded integration of expert opinions, improving decision accuracy and reliability in complex financial contexts.

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Defining variance functional in terms of the Choquet integral

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In 1979, Kahneman and Tversky developed Prospect Theory (awarded the Nobel Prize in Economics), which postulates that the valuation of gains and losses is asymmetric [3]. The standard definition of the Choquet integral based on two monotone measures extends the concept of expected value and assigns different degrees of importance to the positive and negative data. The aim of this lecture is to introduce a new generalization of the Choquet integral to a multidimensional setup with respect to two different monotone measures describing uncertainties related to gains and losses. This approach would allow the aggregation of n functions into a single scalar output and would extend many probabilistic concepts, such as covariance and variance. We will confront the newly introduced variance functional with the existing ones that are based on the Choquet-like aggregation [4, 2, 1].

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Smooth associative idempotent functions on finite lattices

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Continuity is often a desired property in many fields of applied mathematics due to its ability to model gradual change, computational stability, etc. However, in the finite case, the notion of continuity is lost. Therefore, on finite chains, smoothness was proposed as one of the alternatives to continuity (see [3]). The most important classes of smooth associative aggregation functions, such as t-norms [3] and uninorms [2] on finite chains, were studied. It was noted that there is no proper smooth uninorm on a finite chain, similarly as it is the case of continuous uninorms on the unit interval.

Recently [7], the concept of smoothness was also extended for finite lattices. In this contribution, we discuss the properties of smooth associative aggregation functions such as t-norms, t-conorms, nullnorms, and uninorms on finite lattices. Specifically, the existence of an idempotent smooth t-norm is ensured if and only if the finite lattice is lower semimodular.

Already in [7] the existence of smooth non-commutative pseudo-t-norms was noted, which may lead to the study of a non-symmetric conjunction and disjunction and thus non-symmetric logic, which strongly differs from the case of a finite chain (see [4]). Observe that for the existence of a smooth t-norm, the structure of a finite lattice plays an important role.

Uninorms and nullnorms were extended to bounded lattices by Karaçal et al. in [5,1]. Unlike in the case of finite chains and the unit interval, we are able to define smooth proper uninorms on certain finite lattices. Moreover, each smooth idempotent uninorm is necessarily a smooth idempotent nullnorm. Due to the characterization of idempotent nullnorms in [6], we are able to discuss the structure of smooth idempotent uninorms and nullnorms in detail.

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Divisibility of Aggregation Functions on Bounded Lattices

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Aggregation as a binary operation on the unit interval has been widely studied in fuzzy set theory. The importance of aggregation functions is made apparent by their wide use, not only in pure mathematics, but also in several applied fields such as operation research, computer and information sciences, economics, and social sciences.

A *bounded lattice* $(L, \leq_L, 0, 1)$ is a structure in which for every $x, y \in L$, both $x \wedge y$ and $x \vee y$ exist in L, and has the top and bottom elements 1 and 0, respectively [7]. Recent studies have focused mostly on aggregation functions on lattices [11,6,5,3,2,10]. This is a more general case and covers the previous framework.

Divisibility is a fundamental concept in the study of binary operations, specifically aggregation functions, providing insight into their structural properties. Historically, divisibility has been primarily investigated within chains or bounded lattices for t-norms, t-conorms [8,4,1], and later for some other operations such as associative aggregation functions [9]. Despite these developments, a comprehensive research of divisibility for more general structures, specially on bounded lattices, is absent.

This work aims to extend the notion of divisibility for more general functions than just for t-norms and t-conorms, considering it on bounded lattices and mostly, in commutative frameworks. We start with point-wise divisibility:

Definition 1 Let $* : L^2 \to L$ be an operation. If $*$ is monotone in the right (left) argument, we say $*$ is right (left) point-wise divisible at $y \in L$ if for all $x \in L$ with $x \in [y * 0, y * 1]$ ($x \in [0 * y, 1 * y]$) there exists $z \in L$ such that

$$y * z = x \quad (z * y = x).$$

Monotone operation $*$ is point-wise divisible at y if it is point-wise divisible both left and right.

Based on this definition, we introduce the definition of divisibility for any monotone operation on bounded lattices.

Definition 2 Let $* : L^2 \to L$ be an operation. Then we say $*$ is right (left) divisible if it is monotone in the right (left) argument and for all $y \in L$ it is right (left) point-wise divisible.

Operation $*$ is divisible if it is both right and left divisible.

Besides, considering these given definitions for some aggregation functions, we focus on the divisibility of uninorms and nullnorms on bounded lattices, which was a completely uncovered area even in the case of chains.

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On Combination of T-Norms and T-Conorms in Fuzzy Inference Systems

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Fuzzy inference systems are traditionally used in modeling of different processes, they are also part of decision making and daily applications in technological devices. Fuzzy inference systems have been studied from many angles in, e.g. [1] and [2], and they have been also narrowed down to some specific areas, e.g. risk assessment in [3] which is also in focus of our research.
As part of setting up the fuzzy inference system different input values are evaluated. In this process we consider Mamdani and Sugeno fuzzy inference systems which are widely analyzed in most of researches. The input values are consolidated in the next step, and aggregation operators are applied to find respective output values. Under usual constructions, minimum t-norms or maximum t-conorms are used. We propose to apply combination of t-norms and t-conorms, and assess the impact of such combinations on the outputs of the fuzzy inference systems. The combination of the aggregation operators

had been previously applied in our research for simple aggregation of risk levels, see, e.g. [4], where combination of maximum t-conorm $S_M = \max(x_1, ..., x_n)$ and Łukasiewicz t-conorm $S_L = \min(1, \sum_{i=1}^{n} x_i)$ was proposed in the format of $S_C = \max(x_1, ..., x_k, \min(1, \sum_{j=k+1}^{n} x_j))$. While introducing combination of the aggregation operators in fuzzy inference systems, we also consider product t-conorm $S_P = \sum_{i=1}^{3} x_i - x_1x_2 - x_1x_3 - x_2x_3 + x_1x_2x_3$ (in case of three input values) and combine the product t-conorm S_P with maximum t-conorm S_M using arithmetic average of both t-conorms: $(S_P + S_M)/2$. As the next step we also analyze the application of family of Hamacher t-norms and how they can be combined with other t-norms and t-conorms in construction of outputs. It should be admitted that in case of simple aggregation by means of t-norms and t-conorms the input values can be considered as independent while in case of fuzzy inference systems the input values are interdependent and they jointly impact the output. Therefore we are particularly focusing on how this interdependence correlates with application of selected t-norms and t-conorms.

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Generating Interpretable and User-elicited Fuzzy RDF Properties

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This contribution introduces a novel, formally-grounded framework for interpretable link generation where users can direct the inference process. We leverage intercontextual bonds from Fuzzy Formal Concept Analysis [6] to synthesize new fuzzy RDF properties by composing evidence from different contexts, guided by explicit domain knowledge. Our framework provides a choice between complementary inference strategies: a benevolent bond for optimistic evidence aggregation and two distinct rigorous bonds for pessimistic, high-certainty validation [2,4,3]. We establish the formal soundness of our approach with key theoretical results, including duality theorems and a proof that our rigorous constructions are the greatest solutions to specific systems of fuzzy relational equations [5,1]. This provides a robust computational model and guarantees full traceability for every inferred link. Through detailed case studies, we demonstrate the framework’s superior interpretability and its capacity for auditable, user-guided knowledge generation, offering a transparent alternative to opaque Knowledge Graph Completion models.

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Fuzzy entropy and topological entropy of fuzzy dynamical systems

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In this contribution, we summarize some results on topological entropy of fuzzy dynamical systems [2], taken from the theory of "crisp" discrete dynamical systems. In the next part, we also explore the notion of fuzzy entropy when different definitions of fuzzy compactness are considered. Inspired by recent Akin's work [3], we plan to provide some bridge results between the two directions mentioned above. We, for instance, state that topological entropy agrees with fuzzy topological entropy when the notions are defined with the help of Lowen's definition of fuzzy compactness [1]. The particular case of interval maps is also discussed, proving that the fuzzy entropy also agrees with the topological entropy of the Zadeh fuzzification of the crisp map on the set of convex and normal fuzzy sets. The talk is based on a joint work with J. S. Cánovas, Technical University of Cartagena, Spain.

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Operations on fuzzy variables with constraint-based interactivity

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The importance of studying interactivity of fuzzy variables has been acknowledged since their introduction by Zadeh in 1975 [1]. However, there still is insufficient knowledge on how to describe the joint possibilistic distribution of a higher-dimensional fuzzy variable in terms of its marginal distributions. When propagating uncertainty through operations on fuzzy variables, the need for a suitable calculus for interactivity is motivated by the natural desire to prevent information deficiency. In general, computational methods taking into account interactivity yield more specific results than interactivity-agnostic methods [2]. We discuss operations on n-dimensional fuzzy variables ($n \geq 2$) in settings where the marginal fuzzy variables are related by functional constraints. Roughly, the more restrictive the constraint, the larger the disparity between interactive and agnostic results.

A common motivating example is the difference $A - B$, where A and B are identical fuzzy variables. Extending the difference for real numbers, the difference of two identical fuzzy variables should be (the singleton) zero. This result is reached using interactive computations. However, if A is non-constant, agnostic computations produce an information loss. This equality constraint can be broadened to a functional constraint $B = f(A)$. In this case, for a function G, we also investigate the composition $G(A, f(A))$. Similarly, we can consider a linear inequality constraint $B \leq A$, and the more general $B \leq f(A)$.

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Admissible pairs of quasi-linear means and related families

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The study of admissible orders on the family of closed subintervals of $[0, 1]$ – that is, total orders refining the Kulisch-Miranker order – originates from the work of Bustince et al. [4]. By establishing a consistent framework for comparison and ranking, admissible orders enable the effective analysis of interval-valued data, which frequently arise in uncertain or imprecise domains such as decision making and classification [7, 6, 3]. An interesting property of an admissible order is that it can be generated by an admissible pair of aggregation functions, acting on the endpoints of intervals [4, 1], a prominent example being the pair of weighted arithmetic means with distinct weights [4, 1]. In this contribution, we develop a general framework for constructing admissible pairs of aggregation functions, thereby extending the range of known examples. Our approach enables a unified verification of admissibility across several important classes of functions, including quasi-linear means, Archimedean t-norms and t-conorms, and selected strictly Schur-convex or Schur-concave functions. These studies are essential for constructing interval-valued operators defined with respect to an admissible order generated by an admissible pair of aggregation functions [5, 2]. Moreover, we explore the relationship between admissible orders generated by pairs of aggregation functions and the (α, β)-order, identifying the circumstances under which these two notions do not coincide.

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A note on d-variate polynomial copulas

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The hardware used to compute the values of real-valued functions is based only on addition, subtraction, and multiplication, and is therefore limited to polynomial operations. The values of polynomials are quite easy to compute, and polynomials have very favourable properties when it comes to approximate real-valued functions. Therefore, it makes sense to study polynomial d-copulas in detail to take advantage of these properties. As an example, observe that d-copulas which are polynomials of degree n are absolutely continuous and their densities are polynomials of degree $n-d$.

A distinguished class of polynomial copulas is the family of EFGM copulas [7,6,3,5,4]. The smallest possible degree of a polynomial d-copula is just d (for the independence copula Π). There are no proper $(d+1)$-polynomial d-copulas, and the only proper $(d+2)$-polynomial d-copulas are EFGM copulas, given by

$$C(x_1,\dots,x_d) = \left(1+\sum_{1\le i<j\le d}\alpha_{ij}(1-x_i)(1-x_j)\right)\prod_{i=1}^{d}x_i,$$

where $\sum_{1\le i<j\le d}|\alpha_{ij}|\le 1$.

It should be stressed that polynomial d-copulas of fixed degree $n>d+2$ are completely known only in the case $d=2$, where the corresponding polynomials of degree 5 are characterized by three real constants (a,b,c), see [2].

Recall that d-copulas are functions $C\colon [0,1]^d \to [0,1]$ which are d-increasing and satisfy the boundary conditions (0 is the zero element, and 1 is the neutral element). Formally, one can also define 1-copulas, in which case the only 1-copula is the identity function $C(x)=x$.

Based on these boundary conditions, we can formulate a necessary condition for the form of a polynomial d-copula. Considering 2-copulas only, we show that the Darsow product $C*D$ of two 2-copulas C and D is a polynomial copula whenever both C and D are polynomials. In this contribution, we mainly focus on polynomial 3-copulas of degree 6, where we introduce several interesting subclasses, such as Bernstein polynomial copulas, see [3](Theorem 4.1.10) and [1](Theorem 3.1). An interesting approach for constructing polynomial copulas of higher dimensions by means of lower-dimensional copulas will also be presented. For example,

$$C(x_1, x_2, x_3) = x_1 D(x_2, x_3)$$

defines a ternary polynomial copula of degree $k+1$, where D is an arbitrary binary polynomial copula of degree k.

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Time-dependent fuzzy measures

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Fuzzy measures [1] are non-additive measures that are used in the context of fuzzy set theory. They are set functions that generalize classical additive measures by relaxing the requirement of additivity while preserving important properties such as monotonicity. Fuzzy measures and their connection with aggregation functions have been studied in many works, for instance see [4, 1]. Many real-world systems change over time. To model such systems, the concept of time-dependent fuzzy measures can be considered, allowing the measure assigned to each element to change dynamically with time. Here, we introduce two primary frameworks for time-dependency into fuzzy measures; memoryless and dynamical time-dependent fuzzy measures. In memoryless time-dependent fuzzy measures, the fuzzy measure at each time t is determined directly from current input data without reference to the past. Such models are simple and suitable for rapidly changing or unstructured environments.

A more advanced approach incorporates memory by defining time-dependent fuzzy measures as a combination of historical memory and a new measure. This dynamic model is powerful in applications that require adaptation and learning over time.

As Sugeno integrals are an important family of aggregation functions which have a close relation with fuzzy measures, we choose this family to study their time-dependent version. The time-dependent Sugeno integral is a generalization of the classical Sugeno integral, where both the input data and the fuzzy measure may vary with time. It is particularly useful for modeling decision-making or aggregation processes where inputs or evaluations change over time, and the importance of different subsets can also evolve. Note that considering time-dependent Choquet integrals, which are additive in measures, can be interesting too.

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Inner inverses and solutions of interval linear systems

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An important issue in solving fuzzy linear systems is obtaining all formal (algebraic) solutions of interval linear systems (ILS). Finding "the best" interval solution for an interval system of linear equations is demanding, known to be NP-hard. The concept of generalized inverses plays a central role in solving systems of linear equations, it turns out that it plays a central role in obtaining formal solutions of ILS with precise coefficient matrices of arbitrary size.

A characterization of formal solutions of ILS, based on inner inverses, i.e. $\{1\}$-inverses of their precise coefficient matrices, was presented in [1]. Additionally, in the same paper, a straightforward approach for obtaining the united inner solution set of systems of linear interval equations with real coefficient matrices, their classification and the maximal inner interval estimates are presented. In order to explore solution sets of formally consistent interval linear systems, with complex coefficient matrices, it is useful to determine all algebraic solutions of complex interval linear system (CILS), $A[\mathbf{X}] = [\mathbf{B}]$, where $A \in \mathbb{C}^{m\times n}$ is a complex $m \times n$ matrix and $[\mathbf{B}] \in \mathbb{IC}$ is given interval m-dimensional vector of complex intervals. A generalization of the method presented in [1], combined with the methodology of solving the rectangular fuzzy complex linear systems of size $n \times n$ presented in [2], is a promising approach for solving the rectangular fuzzy complex linear systems of arbitrary size, by choosing any of $\{1\}$-inverses of its complex coefficient matrix.

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A Fuzzy Logic Approach to Soft Classification

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This work continues the research of studies [2] - [1] and presents a fuzzy logic-based framework for addressing soft classification problems, where the goal is to determine the degree to which an object belongs to different classes rather than making a crisp assignment. The proposed approach extends the traditional k-Nearest Neighbors (k-NN) algorithm by introducing fuzzy equivalence relations as a measure of similarity between objects. These relations are defined using different t-norms, including the Łukasiewicz, product, and Hamacher t-norms, which govern the transitivity property of the fuzzy relations and influence the structure of the similarity space.

To construct these fuzzy equivalence relations, the method employs the concept of an additive generator, which provides a systematic way to derive t-norm-based transitivity measures. For the final classification stage, aggregation operators are applied to combine similarity values obtained from the k nearest neighbors, resulting in a set of class membership degrees that reflect the uncertainty inherent in the data.

The proposed fuzzy k-NN approach is evaluated through a comparative analysis with standard classification methods, including conventional k-NN and Logistic Regression. The results demonstrate that incorporating fuzzy logic enhances the flexibility and robustness of the classification model, particularly in cases involving overlapping class boundaries or noisy data. Moreover, the use of fuzzy equivalence relations based on different t-norms allows for a deeper understanding of how similarity structures affect classification performance.

Overall, this study highlights the potential of fuzzy logic as an effective tool for soft classification, providing a more interpretable and adaptable alternative to traditional crisp classification methods.

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Direct-optimal systems of attribute implications in fuzzy Formal Concept Analysis

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The computation of closures via systems of implications is defined as a recursive method until a fixed point is reached. Thus, two main research lines arise, either reducing the size of the system of implications so that the computation is as efficient as possible, or ensure conditions so that the fixed point is reached in only one iteration. In the crisp case, the latter research line, namely the direct bases of implications were studied by Rodríguez-Lorenzo et al. [1].

This contribution studies direct bases of implications in the fuzzy setting. The directness of systems allows a quick computation of the closure operator in cases such as Fuzzy Formal Concept Analysis. Characterizing these properties in algebraic terms is deeply linked to Simplification Logic. This is a proper extension of the study in the crisp case since the Fuzzy Attribute Simplification Logic is fundamentally different [3].

First, the theoretical background is considered, in particular, the algebraic conditions that characterize directness are described. Then, some thoughts on algorithms to provide direct systems are also considered.

In addition to the results provided in [2], some ideas and algorithms are proposed to obtain a direct-optimal basis of implications, joining the efforts of both research lines, that is, the closure is obtained in one single iteration and this iteration has the least computational cost.

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Unsharp residuation in posets

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Starting from Boole's early investigations on the laws of human thought, numerous logics of definite importance to contemporary mathematics have been lattice or semilattice based. Paramount examples include classical logic, intuitionistic logic, and many-valued logics, whose algebraic counterparts such as Boolean algebras, Heyting algebras, MV algebras, MTL algebras, and residuated lattices are based on lattice reducts. However, there exist propositional logics where logical connectives need not be defined everywhere. A notable example is the logic of quantum mechanics, where the disjunction of two propositions may not exist when these propositions are neither orthogonal nor comparable. This motivated researchers to consider orthomodular posets instead of orthomodular lattices.

An alternative approach, called "unsharp," was first mentioned by Giuntini and Greuling and later elaborated for several logical connectives. Rather than having undefined values of propositional connectives, this approach allows for multiple non-comparable possibilities for results, where it is uncertain which can be determined as superior. This paper [1] demonstrates that such reasoning can be extended to "fuzzy-like" logics based on posets, proving that any finite poset can serve as the foundation for an unsharp logic.

We introduce the concept of unsharp operators in bounded posets satisfying the Ascending Chain Condition (ACC), demonstrating that these operators form an unsharp residuated pair. The set of all antichains of such a poset constitutes a Heyting algebra. We then develop the theory of uni-posets, structures constructed from posets endowed with multiplication operations. Under specific conditions, these specialize into binary sup-algebras or quantales. We establish categorical properties, showing that the category of uni-posets has a reflector in the category of binary sup-algebras, though this reflector does not preserve monomorphisms. Finally, we formally introduce muni-posets and unsharp residuation for posets with ACC, proving that these structures also form a category with a reflector in the category of binary sup-algebras satisfying ACC. Throughout, we provide numerous examples illustrating these concepts in both modular and non-modular lattices.

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Probabilistic Short-Term Rain Forecasting with Radar Images

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Short-term Rain forecasting, also known as nowcasting, is an essential tool to anticipate potential water-related risks like floods. The reflectivity weather radar is a ground sensor that emits radio waves that are reflected by water rain. It generates a precipitation map that estimates the amount of rain currently falling in a circular area. We employ probabilistic diffusion models to make rain forecasts of the future 90 minutes of radar images (based on [2]) with the goal to anticipate heavy rainstorms that can become a risk for society. Despite forecasts achieving good accuracy with metrics like Fraction Skill Score (FSS) [1], the chaotic behaviour of the physics of the atmosphere is unpredictable, so single forecasts are unreliable by themselves, as they only capture one possible scenario. However, if we combine different forecasting models that can account for different possible scenarios, we can turn the individual unreliable predictions into a robust and reliable forecast. To generate reliable forecasts with practical use, it is necessary to model the uncertainty that comes with every single forecast in a way that a non-expert human can interpret and use to make decisions but also allowing advanced users to see more technical information. To make that, we employ image fusion techniques adapted to this problem to generate a highly probable rain scenario for general use, a scenario that highlights low probability high intensity rain scenarios that can realistically occur, and detailed precipitation probability maps for expert users. The image fusion process currently utilizes forecasts of models that generally achieve general good FSS with every model having the same amount of weight. However, we are studying the possibility of assigning different weights to different models depending on input data patterns, like the speed of the storm, the presence of certain types of rain cells and their distribution, etc.

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From Fuzzy Partitions to Reproducing Kernel Hilbert Spaces

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In this contribution, we show that one of the possible approaches to the definition of a fuzzy partition [5,4,3] (especially in multidimensional space) is to relate this concept to a positive definite (p-d) kernel and thereby define the fuzzy partition elements as its (p-d kernel) feature maps. On this theoretical basis, we can use the close relationship between p-d kernels and reproducing kernel Hilbert spaces (RKHS) and their feature maps.

A constructive approach from a fuzzy partition through the kernel associated with it to the unique Hilbert space, for which this kernel is reproducing, is based on the Moore-Aronszajn theorem [2]. It can be shown in more detail that the definition of a fuzzy partition can be uniquely related to the p-d-symmetric kernel and thus defines a unique RKHS whose only reproducing kernel is it.

The importance of the proposed theoretical construction lies in its connection with the so-called Representer Theorems [1], which help to reduce the general problem of machine learning to algorithms implemented on computers.

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Alternative Statistical Moments under a Choquet Integral Framework

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There exist multiple types of statistical moments, each tailored to address specific challenges in data analysis. Standard score moments are linked to statistical efficiency, T-moments to robust statistical theory, M-moments to estimating functions, with L-moments being their more robust version, and TL-moments are designed to handle outliers. However, these concepts often remain disconnected, with limited tools for systematic comparison.

This contribution introduces a unifying mathematical framework, the Choquet integral quadruplet, which generalizes statistical moments using non-additive set functions. A central question addressed is how standard fuzzy moments relate to alternative additive moments (L-, TL-, T-, and PW-moments), and whether they coincide under certain structural conditions. To formalize this, we define the k-th U-moment of a random variable X, where $U \in$

$\{M, L, TL, T, \dots\}$ denotes the specific moment class. The corresponding variance is defined as $U\text{-VAR}[X](\theta) := e_2^U(\theta) - \left(e_1^U(\theta)\right)^2$. We show that, under suitable choices of the distortion function, Choquet moments can encompass each of these alternative moment types. In particular, the relationship between standard fuzzy variance and the newly defined U-VAR variance provides new insight into moment and variance estimation in non-additive settings.

The Choquet integral quadruplet framework clarifies relationships between moment types, supports generalization beyond classical assumptions, and enhances the interpretability of robust statistical measures. This contribution focuses on these theoretical foundations and demonstrates the unifying potential of Choquet-based moments through selected examples and comparisons.

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Towards a Solver-Free Best-Worst Method

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Multi-criteria decision-making methods such as Saaty Analytic Hierarchy Process, Thurstone Method, and the Best-Worst Method are widely recognised tools for deriving criteria weights [4], [3], [1]. Although effective, some of these methods require solving optimisation problems, which increases computational effort and limits their wider applicability.

This contribution introduces an extension of the classical Best-Worst Method that simplifies the weighting process. Instead of solving an optimisation task, weights $w_i, i = 1, 2, \dots, n$, are derived solely by assigning integer values from 1 to 9 to the others-to-the-worst indicator. They are then calculated directly according to the formula

$$w_i = \frac{f(x_i) \times x_i}{\sum_{i=1}^{n} f(x_i) \times x_i}.$$

For increasing functions f, the weights are consistent by construction, while in the case of decreasing functions, sufficient conditions must be satisfied to guarantee monotonicity and consistency.

The main advantage of this approach lies in its simplicity and efficiency. It eliminates the need for solvers and objective function evaluation, making it suitable for applications with many criteria or for evaluators lacking optimisation expertise. The proposed extension thus represents a practical and robust alternative that combines methodological soundness with usability and can be applied, for example, in the assessment of pension system sustainability [2].

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On aggregation of Ω-groups and fuzzy subgroups

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In classical algebraic systems, the family of all substructures of a structure is closed under intersection, while unions preserve the given structure only in certain cases. For example, the union of two subgroups is a subgroup if and only if one of them is contained in the other one. A similar problem appears in fuzzy settings, where the aggregation of two fuzzy subgroups seldom results in a fuzzy subgroup. Some exceptions in a few specific situations are described in [1,4,2]. Motivated by these limitations, aggregation processes in specific type of lattice-valued algebraic structures are investigated. The focus of this paper is on Ω-groups, where Ω is a complete distributive lattice ([5]).

Both the aggregation of equivalence relations compatible with the algebraic structure and the aggregation of fuzzy groups are investigated, using a lattice-valued approach with the minimum T-norm and the cut-based interpretation. The problem is observed in reverse: instead of constructing an aggregation directly, the resulting Ω-group that should represent the aggregate of two given Ω-groups is first identified, and then the conditions under which an aggregation operator can produce such a structure are analyzed. The structural properties of Ω-groups are characterized, and the preservation of these properties under aggregation is studied. Particular emphasis is placed on possibility of using discrete fuzzy integrals as aggregation operators for this type of problem. Since the approach is lattice-valued, the assumption is that the aggregation operators are compatible with the lattice structure [3].

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The Application of the Choquet integral in Fuzzy Cognitive Maps

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The Choquet integral is a flexible aggregation tool that enables the consideration of nonadditive measures, thereby effectively modeling the nonlinear relationships present in complex systems. Since traditional additive models often fail to adequately handle the interactions between interconnected factors, the advantage of the Choquet integral lies in its ability to aggregate information in a weighted manner, accounting for intricate dependencies among inputs. Fuzzy cognitive maps are models that describe the internal relationships of systems as dynamic networks, while learning algorithms are adaptive computational methods that allow these models to automatically refine their parameters based on incoming data. This is particularly important for fuzzy cognitive maps, which are initially based on expert knowledge but require continuous modification due to dynamically changing environments. Through learning algorithms, the weights and connections within the maps can be automatically optimized, thereby improving the predictive accuracy and adaptability of the model. The application of the Choquet integral in fuzzy cognitive maps creates new opportunities for more accurate modeling of system behavior. By explicitly handling non-additive effects and complex interactions, decision-support systems can be enhanced in terms of performance. This is particularly advantageous in cases where traditional weighting and aggregation methods fail to properly represent the complexity of evolving relationships, thereby contributing to increased reliability and adaptability of the models.

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Fuzzy Satisfaction of a Signal Temporal Logic Formula

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Signal Temporal Logic (STL) is a formalism used to specify cyber-physical systems firstly introduced in [1]. It is very expressive and allows avoiding vagueness inherent to natural language. However, its expressiveness can be improved by combining fuzzy satisfaction of the predicates composing an STL formula with arithmetic operations. In this work, we extend the STL formalism to address fuzzy sets as predicates, and introduce membership functions as in [3]. The proposed syntax is defined iteratively as follows:

$$\varphi := \alpha \mid \pi \mid \mu \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \,\mathcal{U}_I\, \varphi_2 \qquad (1)$$

where φ is an STL formula, $\alpha \in [0,1]$ a satisfaction score, $\pi : \mathbb{R}^n \to [0,1]$ a set predicate, $\mu : \mathbb{R}^n \to [0,1]$ a membership function used to combine predicates, $\neg$ the negation operator, $\wedge$ the conjunction operator, and $\mathcal{U}_I$ the until operator on the time interval I, stipulating that φ_2 must be satisfied at a time $t' \in I$, and $\forall t'' < t', \varphi_1$ is satisfied.

Finally, a satisfaction score is computed to represent the possibility of satisfaction of the formula. The interest of the approach is twofold:

(i) it can deal with reachable sets as traces (see [2]) and also fuzzy specifications (using predicate π). Thus, uncertainties naturally coming from real systems can be taken into account and the verification process is reliable.

(ii) it brings simpler formulas and more expressiveness by authorizing relaxed constraints (using membership function μ). For example, it is possible to specify stronger constraint than others in the same formula, or accept that a part of the specifications are missed.

The use of interval analysis to build up the reachable sets and compute the results provides a guarantee on the obtained satisfaction score: a score of 0 or 1 means that the formula is violated or satisfied, respectively. In addition, every score strictly included in the $[0, 1]$ interval indicates a degree of possibility that the formula is satisfied, instead of a less informative result as "uncertain".

Finally, an example of system verification is proposed to showcase the interest of the method: different fuzzy satisfaction scores are computed for a hybrid system following waypoints, with a formula allowing to miss a waypoint but not two in a row.

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On the Behaviour of Morphological Operators under Morphisms in the Category of L-Fuzzy Sets with a Structuring Element

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Mathematical morphology has its origins in geological problems centered on the processes of erosion and dilation. In its classical form, it can be viewed as the transformation of an object A by means of a chosen structuring element S. These transformations rely on translations of S, defined through the additive group structure. In erosion $\mathcal{E}_S$, the transformation reduces A by removing parts determined by S, whereas in dilation $\mathcal{D}_S$, the transformation expands A by adding shifted copies of S. The successive application of dilation and erosion leads to the closing operator $\mathcal{C}_S$, while their reversed order gives the opening $\mathcal{O}_S$. Opening and closing provide two alternative forms of smoothing the chosen A.

Morphological operators are widely used in image processing and analysis to enhance images or emphasize specific regions, with applications in fields such as medical diagnostics, biology, and geology [2], [1], [4].

Among the first works on fuzzy mathematical morphology were papers by De Baets and coauthors [6], [5]. The paper by Nachtegael and Kerre [3] contains a detailed survey of classical fuzzy mathematical morphology in R^n in the first period of its development.

Following the "classical" approach to fuzzy mathematical morphology based on an additive group (X, +, 0) and corresponding families of L-fuzzy sets L^X enriched with the structuring element $S \in L^X$, we introduce the operators of dilation $\mathcal{D}_S : L^X \to L^X$ and erosion $\mathcal{E}_S : L^X \to$

L^X as well as the derived operators of opening $\mathcal{O}_S : L^X \to L^X$ and closing $\mathcal{C}_S : L^X \to L^X$. We study the behaviour of these operators under the mapping $f : (L^X, S) \to (L^Y, T)$ induced by morphisms $f : (X, +_X, 0_X) \to (Y, +_Y, 0_Y)$ of the additive group category, ensuring the preservation of structuring-element relations. We show that, under certain conditions, primarily reflecting the internal relationship between the structuring elements of the spaces, these mappings preserve (in the sense of not diminishing) all fuzzy morphological operators.

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On modifications of clustering algorithms based on fusion methods

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In big data era, there is natural need to handle with information efficiently, structurally, and in real time. To achieve the aim, among other approaches as thresholding, edge detection and region extraction, several clustering methods were introduced and developed within corresponding notion framework. In [1], [2], [3], two types of them are described, studied and improved, namely classical and fuzzy. Applying them leads to a crisp or a fuzzy partition of the data objects set. The resulting information is not so complex and such that it is better to understand.

Referring to new results obtained in fuzzy mathematics and aggregation theory field of study ([4], [5], [6], [7]), the contribution goal is to describe how some crucial algorithm tools can be replaced by new ones. For example, to state centroids, a suitable fusion method can be used (in sense of [7]). It need not to be applied only Euclidian distance to measure data object distances, there is possible to hire pseudometrics, similarity/dissimilarity functions or moderate deviation functions. Instead of K-means and (fuzzy) C-means one can employ suitable aggregation functions.

The proposed algorithm alternatives are tested on intrusion detection data sets composed of multi-class network traffic features representing normal behavior and several types of cyberattacks. The resulting comparison is included and discussed.

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A note on the associativity of functions induced by pair-orders

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A pair-order is the couple $(\leq_1, \leq_2)$ of reflexive and transitive relations (called also quasi-orders) that satisfy the so-called pair-antisymmetry, i.e., for any x, y, if $x \leq_1 y$ and $y \leq_2 x$, then $x = y$. For any idempotent binary function F defined on a set X it is possible to introduce pair of relations (R_1, R_2), which are defined as $(x, y) \in R_1$, iff $F(x, y) = x$ and $(x, y) \in R_2$, iff $F(y, x) = x$. If F is also associative then (R_1, R_2) form a pair-order [2]. We say that an idempotent function is induced by a pair order $(\leq_1, \leq_2)$, if its induced pair of relations is equal to $(\leq_1, \leq_2)$.

We will present results concerning the associativity of idempotent functions induced by pair-orders. Each idempotent (internal), associative function can be represented by a (linear) pair-order, however, there are pair-orders that induce idempotent functions that are not associative. Previously, the associativity of a function induced by a pair-order was studied only for monotone functions defined on a chain. We provide some conditions for associativity of functions induced by pair-orders, defined on general sets.

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An Implicative Fuzzy Rule System for River Water Quality Assessment Based on Expert Opinion

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This work models a river water-quality assessment system intended for subsequent use in decision-making problems. It addresses situations in which decisions rely on qualitative expert assessments rather than precise statistical data. To handle the inherent uncertainty, we construct an implicative fuzzy rule system that enables modeling directly from such qualitative inputs.

The core of our methodology is a fuzzy-modeling approach for assessing river water quality based on expert opinion. For each river, a fuzzy set is defined to represent the degree to which the water quality can be considered good, based on physico-chemical measurements. The evaluation system is built upon fuzzy rules and an implicative approach enriched with linguistic modifiers. This design is crucial for ensuring the coherence of the system [2] and allows us to derive a comprehensive fuzzy representation of water quality. This stands in contrast to our previous work, where we used a conjunctive (Mamdani–Assilian) system, in which such consistency was not guaranteed.

The resulting fuzzy water-quality model serves as the basis for a decision-making framework. To operationalize this framework, the model's fuzzy output is linked to a payoff function (e.g., potential fishing profits) and incorporates a binary action variable (act / do not act) [1]. This mechanism connects expert assessments with practical economic outcomes, enabling quantification of the value derived from improved water-quality information. Calculating the final decision value requires defuzzification at multiple stages, for which the Center of Gravity method is employed. Finally, we numerically verify the monotonicity of the complete system to validate its logical consistency.

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Embedding Similarity with Fuzzy Uncertainty in Retrieval Augmented Generation

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Recent developments in Large Language Models (LLMs) have accelerated the adoption of Retrieval-Augmented Generation (RAG) architectures, where textual generation is supported by embedding-based semantic retrieval. Although highly effective, current RAG pipelines rely on representing documents as fixed point-like embeddings in high-dimensional vector spaces. This assumption ignores the intrinsic semantic variability and uncertainty of natural language [2,1].

The work presented in this talk introduces a conceptual framework for interpreting document embeddings as fuzzy sets rather than as single vectors. Within this approach, each document is modeled as a collection of embedding vectors associated with degrees of membership that reflect their contextual contribution to the overall meaning

$$D = \{(e_i, \mu_i) : e_i \in \mathbb{R}^n, \mu_i \in [0,1]\}.$$

This fuzzy interpretation replaces both the traditional document vectorization and similarity-based retrieval stages in standard RAG pipelines. Instead of ranking fragments on the basis of the cosine distance, the proposed formulation treats retrieval as a fuzzy similarity between a query and a fuzzy set of document embeddings.

The presentation will outline the theoretical foundations of this approach, its motivation within fuzzy set theory, and the open methodological challenges it raises, particularly regarding the definition of similarity measures between points and fuzzy sets in embedding spaces. Preliminary experiments on limited textual data suggest that such representations may capture partial relevance and semantic overlap more naturally than conventional embeddings.

Overall, the contribution of this work lies in proposing a fuzzy-theoretic reinterpretation of document embeddings and sketching a hybrid fuzzy-RAG architecture that embeds uncertainty handling into the retrieval process. The ongoing research aims to examine whether fuzzy modeling can offer a principled way to integrate uncertainty reasoning into next-generation retrieval-augmented generative systems.

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